Abstract: The most naive definition of modular linear differential equations (MLDEs) would be linear differential equations whose space of solutions are invariant under the slash action of the weight \( k \) of \( \Gamma_1 = SL_2(\mathbb{Z}) \), where \( k \) is fixed. Then under an analytic condition for coefficients functions and the Wronskians of a basis of the space of solutions of equations, we have (obvious) expressions of MLDEs as:

\[
L(f) = d^n_k(f) + \sum_{i=2}^{n} P_{2i}d^{n-i}_k(f)
\]

where \( P_{2i} \) is a modular form of weight \( 2i \) on \( SL_2(\mathbb{Z}) \) and \( d_k(f) \) is the Serre derivative. (Of course, we could replace \( \Gamma \) as a Fuchsian group of \( SL_2(\mathbb{R}) \) and modular forms \( P_{2i} \) as being meromorphic.) However, the iterated Serre derivative \( d^n_k(f) \) (which is also called “the higher Serre derivation” because this operator preserves the modularity.) is very complicated since it involves the Eisenstein series \( E_2 \).

MLDEs, of course, can be given in the form

\[
L(f) = D^n(f) + \sum_{i=1}^{n} Q_iD^i(f)
\]

where

\[
D = \frac{1}{2\pi i} \frac{d}{d\tau}.
\]

Then it is not easy to know if the equation above is an MLDE except the fact that \( Q_i \) are quasimodular forms. (It seems hopeless that we verify if \( L(f) = 0 \) is a MLDE.) Very recently, Y. Sakai and D. Zagier (my collaborators) found formulas of \( L(f) \) by using the Rankin-Cohen products between \( f \) and \( g_i \). The latter is a modular form of weight \( 2i \), which is a linear function of the differential of \( Q_j \). Moreover, there is an inversion formulas which express \( Q_i \) as a linear function of the differential of \( g_j \). The most important fact is that the order \( n \) and \( n - 1 \) parts are equal to the so-called higher Serre derivative in the sense of Kaneko and Koike, where the group is \( \Gamma_1 \). (It can be proved for any Fuchsian group.)

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