Algebra Seminar

Modular linear differential equations

Kiyokazu Nagatomo
Osaka University

Abstract: The most naive definition of *modular linear differential equations* (MLDEs) would be linear differential equations whose space of solutions are invariant under the weight k slash action of $\Gamma_1 = SL_2(\mathbb{Z})$, for some k. Then under an analytic condition for coefficients functions and the Wronskians of a basis of the space of solutions of equations, we have (obvious) expressions of MLDEs as:

$$L(f) \,=\, \mathfrak{d}_k^n(f) + \sum_{i=2}^{n} P_{2i}\mathfrak{d}_k^{n-i}(f)$$

where P_{2i} is a modular form of of weight $2i$ on $SL_2(\mathbb{Z})$ and $\mathfrak{d}_k(f)$ is the *Serre derivative.* (We could replace Γ by a Fuchsian subgroup of $SL_2(\mathbb{R})$ and allow the modular forms P_{2i} to be meromorphic.) However, the iterated Serre derivative $\mathfrak{d}_k^n(f)$ (called a "higher Serre derivation" because as an operator it preserves modulality) is very complicated since it involves the Eisenstein series E_2.
MLDEs, of course, can be given in the form

$$\mathsf{L}(f) \,=\, D^n(f) + \sum_{i=1}^{n} Q_i D^i(f)$$

where

$$D = \frac{1}{2\pi\sqrt{-1}}\frac{d}{d\tau}.$$

Then it is not easy to know if the equation above is an MLDE except the fact that Q_i are quasi-modular forms.
Very recently, Y. Sakai and D. Zagier (my collaborators) found formulas of $\mathsf{L}(f)$ by using the Rankin–Cohen products between f and g_i. This is a modular form of weight $2i$, which is a linear function of the differential of Q_j. Moreover, there are *inversion formulas* which express Q_i as a linear function of the derivatives of g_j. The most important fact is that the order n and $n-1$ parts are equal to the so-called higher Serre derivative in the sense of Kaneko and Koike, where the group is Γ_1. (This holds for any Fuchsian group.)

Finally, the most important nature of my talk is that I will use a **blackboard** instead of **slides**ss.

Tuesday, September 3, 2019, 4:00 pm
Mathematics and Science Center: MSC W303

Mathematics
Emory University