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The Counting Lemma for Regular k-uniform Hypergraphs

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THE COUNTING LEMMA FOR REGULAR k-UNIFORM HYPERGRAPHS

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Abstract. Szemerédi's Regularity Lemma proved to be a powerful tool in the area of extremal graph theory. Many of its applications are based on its accompanying Counting Lemma: *If G is an ℓ-partite graph with $V(G) = V_1 \cup \cdots \cup V_\ell$ and $|V_i| = n$ for all $i \in [\ell]$, and all pairs (V_i, V_j) are ε-regular of density d for $1 \le i < j \le \ell$, then G contains $(1 \pm f_\ell(\varepsilon))d^{\binom{\ell}{2}} \times n^\ell$ cliques K_ℓ, where $f_\ell(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Recently, V. Rödl and J. Skokan generalized Szemerédi's Regularity Lemma from graphs to k-uniform hypergraphs for arbitrary $k \ge 2$. In this paper we prove a Counting Lemma accompanying the Rödl–Skokan hypergraph Regularity Lemma. Similar results were independently and alternatively obtained by W. T. Gowers.

It is known that such results give combinatorial proofs to the density result of E. Szemerédi and some of the density theorems of H. Furstenberg and Y. Katznelson.

1. Introduction

Extremal problems are among the most central and most extensively studied in combinatorics. Many of these problems concern thresholds for properties concerning deterministic structures and have proven to be difficult as well as interesting. An important recent trend in combinatorics has been to consider the analogous problems for random structures. Tools are then sometimes afforded for determining with what probability a random structure possesses certain properties.

The study of *quasi-random structures*, pioneered by the work of Szemerédi [46], merges features of deterministic and random settings. Roughly speaking, a quasi-random structure is one which, while deterministic, mimics the behavior of random structures in certain important points of view. The (quasi-random) combinatorial structures we consider in this paper are *set systems* or *hypergraphs*. We begin our discussion with graphs.

1.1. Szemerédi's Regularity Lemma for graphs.

In the course of proving his celebrated Density Theorem concerning arithmetic progressions, Szemerédi established a lemma which decomposes the edge set of any graph into constantly many "blocks", almost all of which are quasi-random (cf. [24, 25, 47]). In what follows, we give a precise account of Szemerédi's lemma.

For a graph $G = (V, E)$ and two disjoint sets A, $B \subset V$, let $E(A, B)$ denote the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$ and set $e(A, B) = |E(A, B)|$. We also set $d(A, B) = d(G_{AB}) = e(A, B)/|A||B|$ for the *density* of the bipartite graph $G_{AB} = (A \cup B, E(A, B))$.

The concept central to Szemerédi's lemma is that of an *ε-regular pair*. Let $\varepsilon > 0$ be given. We say that the pair A, B is ε-regular if $|d(A, B) - d(A', B')| < \varepsilon$ holds whenever $A' \subset A$, $B' \subset B$, and $|A'| > \varepsilon|A|$, $|B'| > \varepsilon|B|$.

We call a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ an *equitable partition* if it satisfies $|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| < t$; we call an equitable partition *ε-regular* if all but $\varepsilon\binom{t}{2}$ pairs V_i, V_j are ε-regular. Szemerédi's lemma may then be stated as follows.

Theorem 1 (Szemerédi's Regularity Lemma)**.** *Let $\varepsilon > 0$ be given and let t_0 be a positive integer. There exist positive integers $n_0 = n_0(\varepsilon, t_0)$ and $T_0 = T_0(\varepsilon, t_0)$ such that any graph $G = (V, E)$ with $|V| = n \ge n_0$ vertices admits an ε-regular equitable partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ with t satisfying $t_0 \le t \le T_0$.*

Szemerédi's Regularity Lemma is a powerful tool in the area of extremal graph theory. One of its most important consequences is that, in appropriate circumstances, it can be used to imply a given graph contains

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a fixed subgraph. Suppose that a (large) graph is given along with an ε-regular partition $V = V_0 \cup V_1 \cup \ldots \cup V_t$ and let H be a fixed graph. If an appropriate collection of pairs $I_H \subseteq \binom{[t]}{2}$ have each $\{V_i, V_j\}$, $\{i,j\} \in I_H$, ε-regular and sufficiently dense (with respect to ε), one is guaranteed a copy of H within this collection of bipartite graphs $E(V_i, V_j)$, $\{i,j\} \in I_H$. This observation is due to the following well-known fact which may be appropriately called the Counting Lemma.

Fact 2 (Counting Lemma)**.** *For every integer ℓ and positive reals d and γ there exists $\delta > 0$ so that the following holds. Let $G = \bigcup_{1 \le i < j \le \ell} G^{ij}$ be an ℓ-partite graph with ℓ-partition $V_1 \cup \ldots \cup V_\ell$ where $G^{ij} = G[V_i, V_j]$, $1 \le i < j \le \ell$, and $|V_1| = \ldots = |V_\ell| = n$. Suppose further all graphs G^{ij} are ε-regular with density d. Then the number of copies of the ℓ-clique K_ℓ in G is within the interval $(1 \pm \gamma) d^{\binom{\ell}{2}} n^\ell$.*

Unlike Szemerédi's Regularity Lemma, Fact 2 is fairly easy to prove.

1.2. **Extensions of Szemerédi's Lemma to hypergraphs.** Several hypergraph regularity lemmas were considered, in part, by various authors [2, 5, 9, 13, 33]. None of these regularity lemmas seemed to admit a companion counting result (i.e. a corresponding generalization of Fact 2). The first attempt of developing a hypergraph Regularity lemma together with a corresponding Counting Lemma was undertaken in [10]. In that paper, Frankl and Rödl established an extension of Szemerédi's Regularity Lemma to 3-graphs, hereafter called the FR-Lemma (see [20, 6] for an algorithmic version).

Analogously to the feature that Szemerédi's Regularity Lemma decomposes a given graph into an ε-regular partition, the FR-Lemma decomposes the edge set of given a 3-graph into constantly many "blocks", almost all of which are, in a specific sense, "quasi-random". The concept of 3-graph regularity which plays the analogous rôle of the ε-regular pair is, unfortunately, considerably more technical than its graph counterpart. As well, it is not necessary at this time to know this precise definition in order to understand the current Introduction. We therefore postpone precise discussion until later.

Just as Fact 2, the Counting Lemma, is an important companion statement to Szemerédi's Regularity Lemma, most applications of the FR-Lemma require a similar companion lemma - the "3-graph Counting Lemma". Analgously to Fact 2, the 3-graph Counting Lemma estimates the number of copies of the clique $K_\ell^{(3)}$ (i.e., the complete 3-graph on ℓ vertices) contained in an appropriate collection of "dense and regular blocks" within a regular partition provided by the FR-Lemma. This 3-graph Counting Lemma was established in [10] for the special case $K_4^{(3)}$ and subsequently fully established by Nagle and Rödl in [28] (see [32] and [30] for alternative proofs). Unlike the case for graphs, the proof of the 3-graph Counting Lemma in [28] was technical and rather lengthy, suggesting that the effort to fully develop hypergraph regularity methods may not be straightforward.

Recently, Rödl and Skokan [40] established a generalization of the FR-Lemma to k-graphs for $k \ge 3$ (see Section 3.2). We will refer to this lemma as the RS-Lemma. In [39], they also succeeded to prove a companion Counting Lemma in the special case of $K_5^{(4)}$. In this paper,

we prove the k-graph Counting Lemma corresponding to the RS-Lemma.

Our Counting Lemma, the main theorem of this paper, requires some notation. Therefore, we defer its precise statement to Section 2.3 (see Theorem 9).

Last, but not least, we mention that a Regularity Lemma as well as a corresponding Counting Lemma for k-graphs was recently proved by Gowers [17]. It is likely that his approach is different from the one taken in [40] and this paper.

1.3. **Quasi-random hypergraphs.** A related line of research is the study of quasi-random hypergraphs, some topics of which play a crucial rôle in our proof. We feel a few words on quasi-random hypergraphs at this time are appropriate.

Haviland and Thomason [19] and Chung and Graham [3, 4] were the first to investigate systematic properties of quasi-random hypergraphs. In particular, Chung and Graham considered several quite disparate looking properties of random-like hypergraphs of density 1/2 and proved that they are, in fact, equivalent. An important concept in their work is the *deviation* of a hypergraph. It is proved in [3, 4] that for fixed integers $\ell \ge k$, a k-graph of density 1/2 with small deviation contains asymptotically the same number of copies of the clique $K_\ell^{(k)}$ as the random hypergraph of the same density. This result can be viewed as a Counting Lemma for that notion of quasi-randomness.

This research was continued by Kohayakawa, Rödl, and Skokan [23] whose approach was based on the concept of *discrepancy* of a hypergraph. Discrepancy is more compatible with respect to the type of regularity a typical "block" exhibits in a partition obtained from the RS-Lemma. One particularly relevant result in [23] is a 'dense Counting Lemma' for hypergraphs with small discrepancy (cf. Theorem 16 in Section 3.1). Unfortunately, the counting needed to match the RS-Lemma deals with a 'sparse' and more difficult environment. However, the 'dense' ancestor of our result plays an important rôle in this paper.

Our attempt for proving the Counting Lemma (corresponding to the RS-Lemma) is to reduce, in an appropriate sense, the harder sparse case to the easier dense case. Our 'reduction' employs the RS-Lemma itself.

1.4. **Applications of the Regularity Method.** Szemerédi's Regularity Lemma together with its corresponding Counting Lemma, Fact 2, has numerous applications (see [24, 25] for excellent surveys). The FR-Lemma and the companion 3-graph Counting Lemma [28] were exploited in a variety of extremal hypergraph problems (cf. [10, 21, 22, 27, 34, 35, 45]).

We believe that the main result of this paper will enable one to apply the RS-Lemma to a variety of hypergraph problems. Some applications combining the RS-Lemma with the Counting Lemma are already considered in [38]. In particular, a conjecture of Erdős, Frankl and Rödl [7] (see also [10] for similar problems) is confirmed in [38].

Theorem 3. *Let $t \geq k \geq 2$ be fixed integers. Suppose that a k-uniform hypergraph $\mathcal{H}^{(k)}$ on n vertices contains at most $o(n^t)$ copies of $K_t^{(k)}$. Then one can delete $o(n^k)$ edges of $\mathcal{H}^{(k)}$ to make it $K_t^{(k)}$-free.*

Theorem 3 has several interesting consequences which we briefly outline below.

Density Theorems.

(*1*) It was shown in [10] (see also [38]) that Theorem 3 implies Szemerédi's well-known Density Theorem [46] concerning integer subsets without arithmetic progressions of prescribed length (cf. [14, 18] for other alternative proofs).

(*2*) In [45], Solymosi pointed out that Theorem 3 also gives the following multi-dimensional version of Szemerédi's Density Theorem originally due to Furstenberg and Katznelson [15]:

Let k and d be fixed integers. If $S \subseteq [n]^d = \{1, \ldots, n\}^d$ is a subset not containing $z + j[k]^d$ for any $z \in [n]^d$ and $j \in [n]$, then $|S| = o(n^d)$.

(*3*) The following theorem of Furstenberg and Katznelson [16] is also a corollary of Theorem 3 (see [37] for details):

Let q and d be fixed integers and V be an n-dimensional vector space over a finite field of order q. If $S \subseteq V$ is a subset not containing an affine d-dimensional subspace, then $|S| = o(q^n)$.

(*4*) It is also shown in [37] that another theorem of Furstenberg and Katznelson [16] can be deduced from Theorem 3:

Let G be a finite abelian group and let $S \subseteq G^n$ be a subset of the product group of n copies of G. If S contains no coset of a subgroup of G^n which is isomorphic to G, then $|S| = o(|G|^n)$.

In [37], both theorems discussed in (*3*) and (*4*) are derived from a more general density result about modules of finite rings.

It is worth mentioning that so far the only known proofs of the theorems of Furstenberg and Katznelson [15, 16] discussed in (*2*)–(*4*) above involve ergodic theory. The purely combinatorial proofs based on the RS-Lemma and the main result of this paper (or similarly proofs based on the recent results of Gowers) give the first quantitative proofs of those theorems. We make no attempt, however, to give any bounds here.

Combinatorial Number Theory and Geometry.

(*5*) In [43] Solymosi gave an alternative proof of the Balog-Szemerédi Theorem [1] which implies the affirmative answer to a conjecture of Erdős:

For every $\delta > 0$ and integer $t > 3$ there is an n_0 so that the following holds. If $A \subseteq \mathbb{Z}$ contains $\delta|A|^2$ arithmetic progressions of length 3 and $|A| > n_0$, then A contains an arithmetic progression of length t.

Unlike the original proof of Balog and Szemerédi, Solymosi's proof is entirely based on Theorem 3 and does not use the well-known theorem of Freiman [11, 12] (see also [41] for a shorter proof of Freiman's Theorem).

(6) In [44] Solymosi applies Theorem 3 to a geometric problem. Roughly speaking he proves that if the number of incidences between hyperplanes and points in dimension d is "close" to the maximum possible, then there are always "dense" subsets, i.e., large point sets such that any d of them are incident to a hyperplane from the arrangement.

(7) In [38] it was shown that Theorem 3 also implies the affirmative answer to a geometric problem of Székely [26, p.226].

Extremal Hypergraph Results. In [29] the authors give a few applications of Theorem 3. For example:

(8) We give a simple (based on Theorem 3) proof of the following Ramsey-type theorem due to Nešetřil and Rödl [31]:

For every integer $\chi \geq 2$ and every fixed k-uniform hypergraph $\mathcal{F}^{(k)}$ there exists a k-uniform hypergraph $\mathcal{H}^{(k)}$ such that every χ-coloring of the edges of $\mathcal{H}^{(k)}$ yields a monochromatic and induced copy of $\mathcal{F}^{(k)}$.

(9) We also extend Turán-type results from [7, 8, 27] concerning the asymptotic number of labeled hypergraphs not containing any copy of a hypergraph from a fixed family.

We intend to give some further applications of the Regularity Method for hypergraphs in the near future.

Finally, we discuss a way to bridge the methods of this paper with the RS-Lemma to produce a new variant of the regularity lemma for k-uniform hypergraphs in Section 8.

2. Statement of the Main Result

2.1. **Basic notation.** We denote by $[\ell]$ the set $\{1,\dots,\ell\}$. For a set V and an integer $k \geq 1$, let $\binom{V}{k}$ be the set of all k-element subsets of V. A subset $\mathcal{G}^{(k)} \subseteq \binom{V}{k}$ is a *k-uniform hypergraph* on the vertex set V. We identify hypergraphs with their edge sets. For a given k-uniform hypergraph $\mathcal{G}^{(k)}$, we denote by $V\left(\mathcal{G}^{(k)}\right)$ and $E\left(\mathcal{G}^{(k)}\right)$ its vertex and edge set, respectively. For $U \subseteq V\left(\mathcal{G}^{(k)}\right)$, we denote by $\mathcal{G}^{(k)}[U]$ the subhypergraph of $\mathcal{G}^{(k)}$ induced on U (i.e. $\mathcal{G}^{(k)}[U] = \mathcal{G}^{(k)} \cap \binom{U}{k}$). A k-uniform *clique* of *order* j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \geq k$ vertices consisting of all $\binom{j}{k}$ many k-tuples (i.e., $K_j^{(k)}$ is isomorphic to $\binom{[j]}{k}$).

The central objects of this paper are ℓ-partite hypergraphs. Throughout this paper, the underlying vertex partition $V = V_1 \cup \dots \cup V_\ell$, $|V_1| = \dots = |V_\ell| = n$, is fixed. The vertex set itself can be seen as a 1-uniform hypergraph and, hence, we will frequently refer to the underlying fixed vertex set as $\mathcal{G}^{(1)}$. For integers $\ell \geq k \geq 1$ and vertex partition $V_1 \cup \dots \cup V_\ell$, we denote by $K_\ell^{(k)}(V_1,\dots,V_\ell)$ the *complete* ℓ-partite, k-uniform hypergraph (i.e. the family of all k-element subsets $K \subseteq \bigcup_{i\in[\ell]} V_i$ satisfying $|V_i \cap K| \leq 1$ for every $i \in [\ell]$). Then, an (n,ℓ,k)-*cylinder* $\mathcal{G}^{(k)}$ is any subset of $K_\ell^{(k)}(V_1,\dots,V_\ell)$. Observe, that $\left|V\left(\mathcal{G}^{(k)}\right)\right| = \ell \times n$ for an (n,ℓ,k)-cylinder $\mathcal{G}^{(k)}$. Observe that the vertex partition $V_1 \cup \dots \cup V_\ell$ is an $(n,\ell,1)$-cylinder $\mathcal{G}^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $k \leq j \leq \ell$ and set $\Lambda_j \in \binom{[\ell]}{j}$, we denote by $\mathcal{G}^{(k)}[\Lambda_j] = \mathcal{G}^{(k)}\left[\bigcup_{\lambda\in\Lambda_j} V_\lambda\right]$ the subhypergraph of the (n,ℓ,k)-cylinder $\mathcal{G}^{(k)}$ induced on $\bigcup_{\lambda\in\Lambda_j} V_\lambda$.

For an (n,ℓ,j)-cylinder $\mathcal{G}^{(j)}$ and an integer $j \leq i \leq \ell$, we denote by $\mathcal{K}_i^{(j)}\left(\mathcal{G}^{(j)}\right)$ the family of all i-element subsets of $V\left(\mathcal{G}^{(j)}\right)$ which span complete subhypergraphs in $\mathcal{G}^{(j)}$ of order i. Note that $\left|\mathcal{K}_i^{(j)}\left(\mathcal{G}^{(j)}\right)\right|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{G}^{(j)}$.

Given an $(n,\ell,j-1)$-cylinder $\mathcal{G}^{(j-1)}$ and an (n,ℓ,j)-cylinder $\mathcal{G}^{(j)}$, we say an edge J of $\mathcal{G}^{(j)}$ *belongs to* $\mathcal{G}^{(j-1)}$ if $J \in \mathcal{K}_j^{(j-1)}(\mathcal{G}^{(j-1)})$, i.e., J corresponds to a clique of order j in $\mathcal{G}^{(j-1)}$. Moreover, $\mathcal{G}^{(j-1)}$ *underlies* $\mathcal{G}^{(j)}$ if $\mathcal{G}^{(j)} \subseteq \mathcal{K}_j^{(j-1)}\left(\mathcal{G}^{(j-1)}\right)$, i.e., every edge of $\mathcal{G}^{(j)}$ belongs to $\mathcal{G}^{(j-1)}$. This brings us to one of the main concepts of this paper, the notion of a *complex.*

Definition 4 (**(n,ℓ,k)-complex**). *Let $n \geq 1$ and $\ell \geq k \geq 1$ be integers. An (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}}$ is a collection of (n,ℓ,j)-cylinders $\{\mathcal{G}^{(j)}\}_{j=1}^k$ such that*

(a) $\mathcal{G}^{(1)}$ is an $(n,\ell,1)$-cylinder, i.e., $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ with $|V_i| = n$ for $i \in [\ell]$,

(b) $\mathcal{G}^{(j-1)}$ underlies $\mathcal{G}^{(j)}$ for $2 \leq j \leq k$.

2.2. **Regular complexes.** We begin with a notion of density of an (n, ℓ, j)-cylinder with respect to a family of $(n, \ell, j-1)$-cylinders.

Definition 5 (**density**)**.** *Let $\mathcal{G}^{(j)}$ be an (n, ℓ, j)-cylinder and suppose $\boldsymbol{\mathcal{Q}}^{(j-1)} = \{\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}\}$ be a family of $(n, \ell, j-1)$-cylinders. We define the density of $\mathcal{G}^{(j)}$ w.r.t. the family $\boldsymbol{\mathcal{Q}}^{(j-1)}$ as*

$$d\left(\mathcal{G}^{(j)} \middle| \boldsymbol{\mathcal{Q}}^{(j-1)}\right) = \begin{cases} \frac{\left|\mathcal{G}^{(j)} \cap \bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|}{\left|\bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|} & \text{if} \quad \left|\bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We now define a notion of regularity of an (n, j, j)-cylinder with respect to an $(n, j, j-1)$-cylinder.

Definition 6. *Let positive reals δ_j and d_j and a positive integer r be given along with an (n, j, j)-cylinder $\mathcal{G}^{(j)}$ and an underlying $(n, j, j-1)$-cylinder $\mathcal{G}^{(j-1)}$. We say $\mathcal{G}^{(j)}$ is (δ_j, d_j, r)-regular w.r.t. $\mathcal{G}^{(j-1)}$ if whenever $\boldsymbol{\mathcal{Q}}^{(j-1)} = \{\mathcal{Q}_1^{(j-1)}, \dots, \mathcal{Q}_r^{(j-1)}\}$, $\mathcal{Q}_s^{(j-1)} \subseteq \mathcal{G}^{(j-1)}$, $s \in [r]$, satisfies*

$$\left|\bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \geq \delta_j \left|\mathcal{K}_j^{(j-1)}\left(\mathcal{G}^{(j-1)}\right)\right|, \text{ then } \quad d\left(\mathcal{G}^{(j)} \middle| \boldsymbol{\mathcal{Q}}^{(j-1)}\right) = d_j \pm \delta_j\,.$$

We extend the notion of (δ_j, d_j, r)-regularity from (n, j, j)-cylinders to (n, ℓ, j)-cylinders $\mathcal{G}^{(j)}$.

Definition 7 (**(δ_j, d_j, r)-regular**)**.** *We say an (n, ℓ, j)-cylinder $\mathcal{G}^{(j)}$ is (δ_j, d_j, r)-regular w.r.t. an $(n, \ell, j-1)$-cylinder $\mathcal{G}^{(j-1)}$ if for every $\Lambda_j \in \binom{[\ell]}{j}$ the restriction $\mathcal{G}^{(j)}[\Lambda_j] = \mathcal{G}^{(j)}\left[\bigcup_{\lambda \in \Lambda_j} V_\lambda\right]$ is (δ_j, d_j, r)-regular w.r.t. to the restriction $\mathcal{G}^{(j-1)}[\Lambda_j] = \mathcal{G}^{(j-1)}\left[\bigcup_{\lambda \in \Lambda_j} V_\lambda\right]$.*

We sometimes write (δ_j, r)-regular to mean $\left(\delta_j, d\left(\mathcal{G}^{(j)} \middle| \mathcal{G}^{(j-1)}\right), r\right)$-regular for cylinders $\mathcal{G}^{(j)}$ and $\mathcal{G}^{(j-1)}$.

Finally, we close this section of basic definitions with the central notion of a regular complex.

Definition 8 (**$(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex**)**.** *Let vectors $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$ of positive reals be given and let r be a positive integer. We say an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ is $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular if:*

- *(a) $\mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{G}^{(1)}$ and*
- *(b) $\mathcal{G}^{(j)}$ is (δ_j, d_j, r)-regular w.r.t. $\mathcal{G}^{(j-1)}$ for $3 \leq j \leq k$.*

2.3. **Statement of the Counting Lemma.** The following assertion is the main theorem of this paper.

Theorem 9 (Counting Lemma)**.** *For all integers $2 \leq k \leq \ell$ the following is true: $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ and there are integers r and n_0 so that, with $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $n \geq n_0$, whenever $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex, then*

$$\left|\mathcal{K}_\ell^{(k)}\left(\mathcal{G}^{(k)}\right)\right| = (1 \pm \gamma) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,.$$

For given integers k and ℓ we shall refer to this theorem by $\mathbf{CL}_{k,\ell}$.

Observe from the quantification $\forall \gamma, d_k\ \exists \delta_k\ \forall d_{k-1}\ \exists \delta_{k-1}\ \dots\ \forall d_2\ \exists \delta_2$, the constants of Theorem 9 can satisfy $\delta_h \gg d_{h-1}$ for any $3 \leq h \leq k$. In particular, the hypothesis of Theorem 9 allows for the possibility that

$$\gamma, d_k \gg \delta_k \gg d_{k-1} \gg \delta_{k-1} \gg \ldots \gg d_h \gg \delta_h \gg d_{h-1} \gg \ldots \gg d_2 \gg \delta_2. \tag{1}$$

Consequently, the Counting Lemma includes the case when complexes $\{\mathcal{G}^{(h)}\}_{h=1}^k$ consists of fairly sparse hypergraphs. It seems that this is the main difficulty in proving Theorem 9.

2.4. **Generalization of the Counting Lemma.** The main result of this paper, Theorem 9, allows us to count complete hypergraphs of fixed order within a sufficiently regular complex. For some applications, it is more useful to consider slightly more general lemmas.

The first generalization enables us to estimate the number of copies of an arbitrary hypergraph $\mathcal{F}^{(k)}$ with vertices $\{1, \dots, \ell\}$ in an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ satisfying that $\mathcal{G}^{(j)}[\Lambda_j]$ is regular w.r.t. $\mathcal{G}^{(j-1)}[\Lambda_j]$ whenever $\Lambda_j \subseteq K$ for some edge K of $\mathcal{F}^{(k)}$. Rather than counting copies of $K_\ell^{(k)}$ in an "everywhere" regular complex, this lemma counts copies of $\mathcal{F}^{(k)}$ in the complex $\boldsymbol{\mathcal{G}}$ satisfying the less restrictive assumptions above. We introduce some more notation before we give the precise statement below (see Corollary 12).

For a fixed k-uniform hypergraph $\mathcal{F}^{(k)}$, we define the j-th shadow for $j \in [k]$ by

$$\Delta_j(\mathcal{F}^{(k)}) = \{J\colon\ |J| = j \text{ and } J \subseteq K \text{ for some } K \in \mathcal{F}^{(k)}\}\,.$$

We extend the notion of a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex to $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex.

Definition 10 (**($\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)}$)-regular complex**). *Let $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$ be vectors of positive reals and let r be a positive integer. Let $\mathcal{F}^{(k)}$ be a k-uniform hypergraph on ℓ vertices $\{1, \dots, \ell\}$. We say an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ with $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ is $\big(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular if:*

- (*a*) *for every $\Lambda_2 \in \Delta_2(\mathcal{F}^{(k)})$, the $(n, 2, 2)$-cylinder $\mathcal{G}^{(2)}[\Lambda_2]$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{G}^{(1)}[\Lambda_2]$,*
- (*b*) *for every $\Lambda_j \in \Delta_j(\mathcal{F}^{(k)})$, the (n, j, j)-cylinder $\mathcal{G}^{(j)}[\Lambda_j]$ is (δ_j, d_j, r)-regular w.r.t. $\mathcal{G}^{(j-1)}$ for $3 \leq j < k$, and*
- (*c*) *for every $\Lambda_k \in \mathcal{F}^{(k)}$, the (n, k, k)-cylinder $\mathcal{G}^{(k)}[\Lambda_k]$ is $(\delta_k, d_{\Lambda_k}, r)$-regular w.r.t. $\mathcal{G}^{(k-1)}$ with $d_{\Lambda_k} \geq d_k$.*

The '$\geq$' in a $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex indicates that we only enforce a lower bound on the densities in the k-th layer of $\boldsymbol{\mathcal{G}}$ (cf. part (c) of the definition). This is the environment which usually appears in applications. We also observe that the Definition 10 imposes only a regular structure on those (m, k, k)-subcomplexes of $\boldsymbol{\mathcal{G}}$ which naturally correspond to edges of $\mathcal{F}^{(k)}$ (i.e., on a subcomplex induced on $V_{\lambda_1}, \dots, V_{\lambda_k}$, where $\{\lambda_1, \dots, \lambda_k\}$ forms an edge in $\mathcal{F}^{(k)}$). We need one more definition before we can state the corollary.

Definition 11 (**partite isomorphic**). *Suppose $\mathcal{F}^{(k)}$ is a k-uniform hypergraph with $V(\mathcal{F}^{(k)}) = [\ell]$ and $\mathcal{G}^{(k)}$ is an (n, ℓ, k)-cylinder with vertex partition $V(\mathcal{G}^{(k)}) = V_1 \cup \dots \cup V_\ell$. We say a copy $\mathcal{F}_0^{(k)}$ of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ is partite isomorphic to $\mathcal{F}^{(k)}$ if there is a labeling of $V(\mathcal{F}_0^{(k)}) = \{v_1, \dots, v_\ell\}$ such that*

- (*i*) *$v_\alpha \in V_\alpha$ for every $\alpha \in [\ell]$, and*
- (*ii*) *$v_\alpha \mapsto \alpha$ is a hypergraph isomorphism (edge preserving bijection of the vertex sets) between $\mathcal{F}_0^{(k)}$ and $\mathcal{F}^{(k)}$.*

Corollary 12. *For all integers $2 \leq k \leq \ell$ and $\forall\gamma > 0$ $\forall d_k > 0$ $\exists\delta_k > 0$ $\forall d_{k-1} > 0$ $\exists\delta_{k-1} > 0$ $\dots$ $\forall d_2 > 0$ $\exists\delta_2 > 0$ and there are integers r and n_0 so that the following holds for $\boldsymbol{d} = (d_2, \dots, d_k)$, $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$, and $n \geq n_0$. If $\mathcal{F}^{(k)}$ is a k-uniform hypergraph on ℓ-vertices and $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ is a $\big(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular (n, ℓ, k)-complex with $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$, then the number of partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ is at least*

$$(1-\gamma) \prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} d_{\Lambda_k} \times n^\ell \geq (1-\gamma) \prod_{h=2}^{k} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times n^\ell\,.$$

Corollary 12 can be easily derived from Theorem 9. Below we briefly outline that proof. The full proof can be found in [42, Chapter 9].

The idea of the proof consists of two basic parts. For $2 \leq j \leq k$, for each $\Lambda_j = \{\lambda_1, \dots, \lambda_j\} \notin \Delta_j(\mathcal{F}^{(k)})$, we replace the (n, j, j)-cylinder $\mathcal{G}^{(j)}[\Lambda_j]$ with the complete j-partite j-uniform system $K_j^{(j)}(V_{\lambda_1}, \dots, V_{\lambda_j})$. Doing so over all $2 \leq j \leq k$ and all $\Lambda_j \notin \Delta_j(\mathcal{F}^{(k)})$ clearly results in an "everywhere" regular complex, let us call it $\boldsymbol{\mathcal{H}}$, whose cliques $K_\ell^{(k)}$ correspond to copies of $\mathcal{F}^{(k)}$ in $\boldsymbol{\mathcal{G}}$.

One now wishes to apply the Counting Lemma, Theorem 9, to the complex $\boldsymbol{\mathcal{H}}$ to finish the job. The only minor technicality in doing so is that, unlike the hypothesis of Theorem 9, the complex $\boldsymbol{\mathcal{H}}$ potentially has, for each $2 \leq j \leq k$, (n, j, j)-cylinders $\mathcal{H}^{(j)}[\Lambda_j]$, $\Lambda_j \in \binom{[\ell]}{j}$, of differing densities. This is handled, however, by "randomly slicing" the (n, j, j)-cylinders $\mathcal{H}^{(j)}[\Lambda_j]$, $\Lambda_j \in \binom{[\ell]}{j}$, into appropriately many pieces of the same density as formally required in Theorem 9. Consequently, we create a series of pairwise $K_\ell^{(k)}$-disjoint complexes $\boldsymbol{\mathcal{H}}_1, \boldsymbol{\mathcal{H}}_2, \dots$, each of which satisfies the hypothesis of the Counting Lemma. Theorem 9 applies to each of the newly created complexes $\boldsymbol{\mathcal{H}}_i$, $i \geq 1$, and so we add the resulting number of cliques to finish the proof of Corollary 12.

The second extension of the Counting Lemma allows us to estimate the number of "non-crossing" copies of a fixed hypergraph $\mathcal{F}^{(k)}$. For that we recall the notion of a homomorphic image of a hypergraph.

Definition 13 (**hypergraph homomorphism**). *Suppose $\mathcal{F}^{(k)}$ and $\tilde{\mathcal{F}}^{(k)}$ are k-uniform hypergraphs. We say $\tilde{\mathcal{F}}^{(k)}$ is an homomorphic image of $\mathcal{F}^{(k)}$ if there exist a surjective map $\vartheta\colon V(\mathcal{F}^{(k)}) \twoheadrightarrow V(\tilde{\mathcal{F}}^{(k)})$ such that for every edge $K \in E(\tilde{\mathcal{F}}^{(k)})$ we have $\vartheta(K) = \bigcup_{v\in K}\vartheta(v) \in E(\tilde{\mathcal{F}}^{(k)})$. We say ϑ is a homomorphism from $\mathcal{F}^{(k)}$ to $\tilde{\mathcal{F}}^{(k)}$.*

In other words, ϑ is a homomorphism from $\mathcal{F}^{(k)}$ to $\tilde{\mathcal{F}}^{(k)}$ if it is an edge-preserving map between the vertex sets of $\mathcal{F}^{(k)}$ and $\tilde{\mathcal{F}}^{(k)}$.

Definition 14 (**ϑ-partite isomorphic**). *Suppose $\mathcal{F}^{(k)}$ is a k-uniform hypergraph with $V(\mathcal{F}^{(k)}) = [\ell]$, $\tilde{\mathcal{F}}^{(k)}$ with $V(\tilde{\mathcal{F}}^{(k)}) = [\tilde{\ell}]$ is a homomorphic image under $\vartheta\colon [\ell] \twoheadrightarrow [\tilde{\ell}]$ and $\tilde{\mathcal{G}}^{(k)}$ is an $(n, \tilde{\ell}, k)$-cylinder with vertex partition $V(\tilde{\mathcal{G}}^{(k)}) = \tilde{V}_1 \cup \dots \cup \tilde{V}_{\tilde{\ell}}$. We say a copy $\mathcal{F}_0^{(k)}$ of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$ is ϑ-partite isomorphic to $\mathcal{F}^{(k)}$ if there is a labeling of $V(\mathcal{F}_0^{(k)}) = \{v_1, \dots, v_\ell\}$ such that*

- *(i) $v_\alpha \in \tilde{V}_{\vartheta(\alpha)}$ for every $\alpha \in [\ell]$, and*
- *(ii) $v_\alpha \mapsto \alpha$ is a hypergraph isomorphism between $\mathcal{F}_0^{(k)}$ and $\mathcal{F}^{(k)}$.*

We now state the second extension of Therorem 9 considered here.

Corollary 15. *For all integers $2 \le k \le \ell$ and $\forall\gamma > 0$ $\forall d_k > 0$ $\exists\delta_k > 0$ $\forall d_{k-1} > 0$ $\exists\delta_{k-1} > 0$ $\dots$ $\forall d_2 > 0$ $\exists\delta_2 > 0$ and there are integers r and n_0 so that the following holds for $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\delta} = \big(\delta_2, \dots, \delta_k\big)$ and $n \ge n_0$.*

Suppose $\mathcal{F}^{(k)}$ is a k-uniform ℓ-vertex hypergraph and $\tilde{\mathcal{F}}^{(k)}$ is a homomorphic image with $\big|V\big(\tilde{\mathcal{F}}^{(k)}\big)\big| = \tilde{\ell}$ under the homomorphism ϑ. If $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^{k}$ is a $\big(\boldsymbol{\delta}, \ge\boldsymbol{d}, r, \tilde{\mathcal{F}}^{(k)}\big)$-regular $(n, \tilde{\ell}, k)$-complex with $\tilde{\mathcal{G}}^{(1)} = \tilde{V}_1 \cup \dots \cup \tilde{V}_{\tilde{\ell}}$, then the number of ϑ-partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$ is at least

$$(1-\gamma)\prod_{\beta\in[\tilde{\ell}]}\frac{1}{|\vartheta^{-1}(\beta)|!}\times\prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\times\prod_{\Lambda_k\in\mathcal{F}^{(k)}} d_{\Lambda_k}\times n^\ell \ge (1-\gamma)\prod_{\beta\in[\tilde{\ell}]}\frac{1}{|\vartheta^{-1}(\beta)|!}\times\prod_{h=2}^{k} d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\times n^\ell\,.$$

Corollary 15 easily follows from Corollary 12. However, the proof is somewhat technical and is given in [42, Chapter 9].

3. Auxiliary results

In this section we review a few results that are essential for our proof of Theorem 9 in Section 4.

3.1. The Dense Counting Lemma.

We recall that Theorem 9 is formulated under the involved quantification $\forall d_k\ \exists\delta_k\ \forall d_{k-1}\ \exists\delta_{k-1}\dots\forall d_2\ \exists\delta_2$ and that the Counting Lemma owes its difficulty in proof to the sparseness arising from this quantification. If the quantification can be simplified so that

$$\min_{2\le j\le k} d_j \gg \max_{2\le j\le k}\delta_j \tag{2}$$

is ensured, then the so-called Dense Counting Lemma (see Theorem 16 below) is known to be true. This was proved by Kohayakawa, Rödl, and Skokan (see Theorem 6.5 in [23]). Observe that (2) represents the 'dense case' in contrast to the 'sparse case' (1), since all densities are bigger then the measure of regularity $\max\delta_j$.

Theorem 16 (Dense Counting Lemma). *For all integers $2 \le k \le \ell$ and any positive constants $d_2, \dots, d_k$, there exist $\varepsilon > 0$ and integer m_0 so that, with $\boldsymbol{d} = (d_2, \dots, d_k)$ and $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^{k-1}$ and $m \ge m_0$, whenever $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ is a $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$-regular (m, ℓ, k)-complex, then*

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big)\right| = (1 \pm g_{k,\ell}(\varepsilon))\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times m^\ell$$

where $g_{k,\ell}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

While the quantification of the main Theorem, Theorem 9, does not allow us to assume (2), Peng, Rödl, and Skokan in [32] used Theorem 16 to prove Theorem 9 for $k = 3$ by reducing the harder 'sparse case' to the easier 'dense case'. This is also the idea of our current proof. The reduction scheme used here, which is entirely different, is somewhat simpler and allows an extension for arbitrary k.

3.2. **The Regularity Lemma.** One of the major tools we use in our proof of Theorem 9 is the recently developed regularity lemma of Rödl and Skokan [40] for k-uniform hypergraphs. Our plan is to apply the regularity lemma to the (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ in the (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ from the hypothesis of Theorem 9. Since $\mathcal{G}^{(k)}$ is ℓ-partite with ℓ-partition $\mathcal{G}^{(1)} = V_1 \cup \ldots \cup V_\ell$, the regularity lemma below is formulated for ℓ-partite hypergraphs.

The regularity lemma of Rödl and Skokan provides well-structured partitions of all complete (n, ℓ, j)-cylinders $K_\ell^{(j)}(V_1, \ldots, V_\ell)$ for $j \in [k-1]$. We later refer to the family of these partitions by $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{k-1})$ is a family of functions which describes the partitions of $\mathscr{R}$ and $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ describes the image sets of $\boldsymbol{\varphi}$. In what follows, we use the language of [40] to give the precise definitions of these concepts.

3.2.1. *Partitions.* Let $V_1 \cup \cdots \cup V_\ell$ be a partition of V with $|V_\lambda| = n$ for every $\lambda \in [\ell]$. Let k be an integer and for every $j \in [k-1]$, let $a_j \in \mathbb{N}$ and let φ_j be a function such that

$$\varphi_j \colon K_\ell^{(j)}(V_1, \ldots, V_\ell) \to [a_j]\,.$$

Note, for every $\lambda \in [\ell]$, mapping φ_1 defines a partition $V_\lambda = V_{\lambda,1} \cup \cdots \cup V_{\lambda,a_1}$, where $V_{\lambda,\alpha} = \varphi_1^{-1}(\alpha) \cap V_\lambda$ for all $\alpha \in [a_1]$. Here, we only consider functions φ_1 such that

$$\big||\varphi_1^{-1}(\alpha) \cap V_\lambda| - |\varphi_1^{-1}(\alpha') \cap V_\lambda|\big| = \big||V_{\lambda,\alpha}| - |V_{\lambda,\alpha'}|\big| \le 1 \tag{3}$$

for every $\lambda \in [\ell]$ and α, $\alpha' \in [a_1]$. Consequently, we have $\lfloor n/a_1 \rfloor \le |V_{\lambda,\alpha}| \le \lceil n/a_1 \rceil$.

Remark 17. *For convenience, we delete all floors and ceilings and simply write $|V_{\lambda,\alpha}| = n/a_1$ for every $\lambda \in [\ell]$ and $\alpha \in [a_1]$.*

Let $\binom{[\ell]}{j}_< = \{(\lambda_1, \ldots, \lambda_j) \in [\ell]^j \colon \lambda_1 < \cdots < \lambda_j\}$ be the set of vectors that naturally correspond to the totally ordered j-element subsets of $[\ell]$. More generally, for a totally ordered set Π of cardinality at least j, let $\binom{\Pi}{j}_<$ be the family of totally ordered j-element subsets of Π. For $j \in [k-1]$ we consider the projection π_j of $K_\ell^{(j)}(V_1, \ldots, V_\ell)$ onto $[\ell]$;

$$\pi_j \colon K_\ell^{(j)}(V_1, \ldots, V_\ell) \to \binom{[\ell]}{j}_<\,,$$

mapping $J \in K_\ell^{(j)}(V_1, \ldots, V_\ell)$ to the totally ordered set $\pi_j(J) = (\lambda_1, \ldots, \lambda_j) \in \binom{[\ell]}{j}_<$ satisfying $|J \cap V_{\lambda_h}| = 1$ for every $h \in [j]$. Moreover, for every $h \in [|J|]$, let $\Phi_h(J) = (x_{\pi_h(H)} = \varphi_h(H))_{H \in \binom{J}{h}}$. In other words, $\Phi_h(J)$ is a vector of length $\binom{|J|}{h}$ and its entries are indexed by elements from $\binom{\pi_j(J)}{h}_<$. For our purposes it will be convenient to assume that the entries of $\Phi_h(J)$ are ordered lexicographically w.r.t. their indices. Observe that for $h > 0$

$$\Phi_h(J) \in [a_h] \times \cdots \times [a_h] = [a_h]^{\binom{j}{h}}\,.$$

We define

$$\boldsymbol{\Phi}^{(j)}(J) = (\pi_j(J), \Phi_1(J), \ldots, \Phi_j(J)).$$

Note that $\boldsymbol{\Phi}^{(j)}(J)$ is a vector with $j + 2^j - 1$ entries. Observe that if we set $\boldsymbol{a} = (a_1, a_2, \ldots, a_{k-1})$ and

$$A(j, \boldsymbol{a}) = \binom{[\ell]}{j}_< \times \prod_{h=1}^{j} [a_h]^{\binom{j}{h}},$$

then $\boldsymbol{\Phi}^{(j)}(J) \in A(j, \boldsymbol{a})$ for every set $J \in K_\ell^{(j)}(V_1, \ldots, V_\ell)$. In other words, to each edge J of cardinality j we assign $\pi_j(J)$ and a vector $(x_{\pi_h(H)})_{H \subset J}$ with each entry $x_{\pi_h(H)}$ corresponding to a non-empty subset H of J such that $x_{\pi_h(H)} = \varphi_h(H)$, where $h = |H|$.

For two edges $J_1, J_2 \in K_\ell^{(j)}(V_1, \ldots, V_\ell)$, the equality $\boldsymbol{\Phi}^{(j)}(J_1) = \boldsymbol{\Phi}^{(j)}(J_2)$ defines an equivalence relation on $K_\ell^{(j)}(V_1, \ldots, V_\ell)$ into at most

$$|A(j, \boldsymbol{a})| \le \binom{\ell}{j} \times \prod_{h=1}^{j} a_h^{\binom{j}{h}}$$

parts. Now we describe these parts explicitly using $(j + 2^j - 1)$-dimensional vectors from $A(j, \boldsymbol{a})$.

For each $j < k$ we define a partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ with partition classes corresponding to the equivalence relation defined above. This way each partition class in $\mathscr{R}^{(j)}$ has a unique address $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$. While $\boldsymbol{x}^{(j)}$ is a $(j + 2^j - 1)$-dimensional vector, we will frequently view it as a $j + 1$-dimensional vector $(\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_j)$, where $\boldsymbol{x}_0 = (x_1, \dots, x_j) \in \binom{[\ell]}{j}_<$ is a totally ordered set and $\boldsymbol{x}_h = (x_\Xi)_{\Xi \in \binom{\boldsymbol{x}_0}{h}_<} \in [a_h]^{\binom{j}{h}}$ for $1 \le h \le j$. For each address $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ we denote its corresponding partition class from $\mathscr{R}^{(j)}$ by

$$\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) = \left\{ R \in K_\ell^{(j)}(V_1, \dots, V_\ell) \colon \boldsymbol{\Phi}^{(j)}(R) = \boldsymbol{x}^{(j)} \right\}.$$

This way we will ensure some structure between the classes from $\mathscr{R}^{(j)}$ and $\mathscr{R}^{(j-1)}$. More formally, for each partition class $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \in \mathscr{R}^{(j)}$ there exist j partition classes $\mathcal{R}_1^{(j-1)}, \dots, \mathcal{R}_j^{(j-1)} \in \mathscr{R}^{(j-1)}$ such that for $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)}) = \bigcup_{h \in [j]} \mathcal{R}_h^{(j-1)}$ we have $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{K}_j^{(j-1)}\big(\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})\big)$. In other words $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ forms an underlying $(j-1)$-uniform j-partite hypergraph of $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})$ consisting of $\binom{j}{j-1}$ classes from $\mathscr{R}^{(j-1)}$. Given $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$ (and the corresponding $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \in \mathscr{R}^{(j)}$) we give a formal definition of $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ below. In fact, for $h < j$ we introduce a notation for the h-uniform j-partite hypergraph $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ which consists of $\binom{j}{h}$ partition classes of $\mathscr{R}^{(h)}$ and satisfies $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{K}_j^{(h)}\big(\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})\big)$.

To that end, we need the following notation. Let $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_0, \dots, \boldsymbol{x}_j) \in A(j, \boldsymbol{a})$ with $\boldsymbol{x}_u = (x_\Upsilon)_{\Upsilon \in \binom{\boldsymbol{x}_0}{u}_<} \in [a_u]^{\binom{j}{u}}$ for $1 \le u \le j$. Given an ordered subset $\Xi \subseteq \boldsymbol{x}_0$ (recall $\boldsymbol{x}_0 \in \binom{[\ell]}{j}_<$) where $|\Xi| = h \le j$. For $1 \le u \le h = |\Xi| \le j$ let

$$\boldsymbol{x}_u^\Xi = (x_\Upsilon)_{\Upsilon \in \binom{\Xi}{u}_<}.$$

be the $\binom{h}{u}$-dimensional vector consisting of those entries of $\boldsymbol{x}_u$ which are labeled with the ordered u-element subsets of Ξ. For each $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_j) \in A(j, \boldsymbol{a})$ and for each $h \in [j]$, we then set

$$\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)}) = \bigcup_{\Xi \in \binom{\boldsymbol{x}_0}{h}_<} \left\{ R \in K_\ell^{(h)}(V_1, \dots, V_\ell) \colon \boldsymbol{\Phi}^{(h)}(R) = (\Xi, \boldsymbol{x}_1^\Xi, \dots, \boldsymbol{x}_h^\Xi) \right\}. \tag{4}$$

Using the language above, the following claim holds.

Claim 18. *For every $j \in [k-1]$ and every $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_j) \in A(j, \boldsymbol{a})$, the following is true.*

(a) *For all $h \in [j]$, $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ is an $(n/a_1, j, h)$-cylinder;*

(b) *$\boldsymbol{\mathcal{R}}(\boldsymbol{x}^{(j)}) = \big\{\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})\big\}_{h=1}^{j}$ is an $(n/a_1, j, j)$-complex.*

To give a precise description of the family of partitions of $K_\ell^{(j)}(V_1, \dots, V_\ell)$, we summarize the notation above in the following Setup in which we work.

Setup 19. *Let $k \le \ell$ and n be fixed positive integers, let $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ be a $(n, \ell, 1)$-cylinder, and $\boldsymbol{a} = \boldsymbol{a}_{\mathscr{R}} = (a_1, a_2, \dots, a_{k-1})$ be a vector of positive integers. Let*

$$A(j, \boldsymbol{a}) = \binom{[\ell]}{j}_< \times \prod_{h=1}^{j} [a_h]^{\binom{j}{h}},$$

and for every $j \in [k-1]$ let $\varphi_j \colon K_\ell^{(j)}(V_1, \dots, V_\ell) \to [a_j]$ be a mapping . Moreover, suppose that φ_1 satisfies (3) *for every $\lambda \in [\ell]$ and α, $\alpha' \in [a_1]$. Set $\boldsymbol{\varphi} = \{\varphi_j \colon j \in [k-1]\}$.*

We now define the family of partitions of $K_\ell^{(j)}(V_1, \dots, V_\ell)$.

Definition 20 (**Partition**)**.** *Given Setup 19, for every $j \in [k-1]$, define apartition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ by*

$$\mathscr{R}^{(j)} = \left\{ \mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \colon \boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a}) \right\}.$$

We also define the family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ and the rank of $\mathscr{R}$ by

$$\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})|.$$

3.2.2. *Polyads.* The ε-regular pair played a central rôle in the definition of a regular partition for graphs (cf. Szemerédi's regularity lemma, Theorem 1). In order to define a regular partition $\mathscr{R}$ for a k-uniform hypergraph, this concept was extended in [40] by introducing *polyads*. Given Setup 19, let $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be the family of partitions as defined in Definition 20. Polyads are $(n/a_1, j, j-1)$-cylinders consisting of selected j members of $\mathscr{R}^{(j)}$ for $j = 2, \ldots, k$. The precise definition of a polyad (which we give below) requires some notation.

Recall that for each edge $J \in K_\ell^{(j)}(V_1, \ldots, V_\ell)$ and $h \in [j-1]$, we defined $\Phi_h(J)$ as the $\binom{j}{h}$-dimensional vector $\Phi_h(J) = (\varphi_h(H))_{H \in \binom{J}{h}}$. We also defined $\pi_j(J)$ to be the totally ordered set $(\lambda_1, \ldots, \lambda_j) \in \binom{[\ell]}{j}_<$ such that $|J \cap V_{\lambda_h}| = 1$ for every $h \in [j]$. We set

$$\hat{\boldsymbol{\Phi}}^{(j-1)}(J) = (\pi_j(J), \Phi_1(J), \ldots, \Phi_{j-1}(J)) .$$

Note that $\hat{\boldsymbol{\Phi}}^{(j-1)}(J)$ is just the projection of $\boldsymbol{\Phi}^{(j)}(J)$ onto its first j vector coordinates. As such, $\hat{\boldsymbol{\Phi}}^{(j-1)}(J)$ is a vector having $j + \sum_{h=1}^{j-1} \binom{j}{h} = j + 2^j - 2$ entries.

We define the set $\hat{A}(j-1, \boldsymbol{a})$ of $(j + 2^j - 2)$-dimensional vectors for $j \in [k-1]$ by

$$\hat{A}(j-1, \boldsymbol{a}) = \binom{[\ell]}{j}_< \times \prod_{h=1}^{j-1} [a_h]^{\binom{j}{h}} .$$

Observe that then $\hat{\boldsymbol{\Phi}}^{(j-1)}(J) \in \hat{A}(j-1, \boldsymbol{a})$ for every edge $J \in K_\ell^{(j)}(V_1, \ldots, V_\ell)$.

Let $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$. We write the vector $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{j-1})$, where $\hat{\boldsymbol{x}}_0 \in \binom{[\ell]}{j}_<$ is an ordered set and $\hat{\boldsymbol{x}}_u = (\hat{x}_\Upsilon)_{\Upsilon \in \binom{\hat{\boldsymbol{x}}_0}{u}_<} \in [a_u]^{\binom{j}{u}}$ for $1 \le u \le j-1$. Given an ordered set $\Xi \subseteq \hat{\boldsymbol{x}}_0$ with $1 \le h = |\Xi| < j$, we set for $1 \le u \le h$

$$\hat{\boldsymbol{x}}_u^{\Xi} = (\hat{x}_\Upsilon)_{\Upsilon \in \binom{\Xi}{u}_<} . \tag{5}$$

For each $j \in [k]$, $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{A}(j-1, \boldsymbol{a})$, and $h \in [j-1]$ we define $\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})$ by

$$\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)}) = \bigcup_{\Xi \in \binom{\hat{\boldsymbol{x}}_0}{h}_<} \Big\{ R \in K_\ell^{(h)}(V_1, \ldots, V_\ell) \colon \boldsymbol{\Phi}^{(h)}(R) = (\Xi, \hat{\boldsymbol{x}}_1^{\Xi}, \ldots, \hat{\boldsymbol{x}}_h^{\Xi}) \Big\}. \tag{6}$$

Note that if $\boldsymbol{x}^{(j)} = \big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ for some $\alpha \in [a_j]$ and $1 \le h < j$, then $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ defined in (4) and $\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})$ defined in (6) are identical.

We also set $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(j-1)}) = \big\{\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{h=1}^{j-1}$. Similarly to Claim 18, we can prove the following.

Claim 21. *For every vector $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{A}(j-1, \boldsymbol{a})$, the following statements are true.*

(a) *For all $h \in [j-1]$, $\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})$ is an $(n/a_1, j, h)$-cylinder;*
(b) *$\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(j-1)}) = \big\{\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{h=1}^{j-1}$ is an $(n/a_1, j, j-1)$-complex.*

In this paper, $(n/a_1, j, j-1)$-cylinders $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ will play a special rôle for $1 < j \le k$ and we will call them polyads.

Definition 22 (**Polyad**). *Given the Setup 19, let $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a family of partitions as defined in Definition 20. Then, for each $1 < j \le k$ and each vector $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$, we refer to the $(n/a_1, j, j-1)$-cylinder $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ as polyad.*

For every polyad $\hat{\mathcal{R}}^{(j-1)}$ there exists a unique vector $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$ so that $\hat{\mathcal{R}}^{(j-1)} = \hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Hence, each polyad $\hat{\mathcal{R}}^{(j-1)}$ uniquely defines an $(n/a_1, j, j-1)$-complex $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(j-1)}) = \big\{\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{h=1}^{j-1}$ such that $\hat{\mathcal{R}}^{(j-1)} = \hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

Remark 23. *For $j = 2$ the set $\hat{A}(1, \boldsymbol{a})$ consists of 2-dimensional vectors $\hat{\boldsymbol{x}}^{(1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1)$, where $\hat{\boldsymbol{x}}_0 = (\lambda_1, \lambda_2) \in \binom{[\ell]}{2}_<$ and $\hat{\boldsymbol{x}}_1 = (\alpha_1, \alpha_2) \in [a_1]^2$. Consequently, a polyad $\hat{\mathcal{R}}^{(1)}(\hat{\boldsymbol{x}}^{(1)})$ is the bipartition $V_{\lambda_1, \alpha_1} \cup V_{\lambda_2, \alpha_2}$*

Every polyad $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ is an $(n/a_1, j, j-1)$-cylinder that is the union of j appropriately chosen partition classes of $\mathscr{R}^{(j-1)}$. We describe these elements using the vectors $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$.

Let $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{A}(j-1, \boldsymbol{a})$ be given. Then, for every $u \in [j-1]$, vector $\hat{\boldsymbol{x}}_u$ can be written as $\hat{\boldsymbol{x}}_u = (x_\Upsilon \colon \Upsilon \in \binom{\hat{\boldsymbol{x}}_0}{u}_<)$, i.e. its entries are labeled by the ordered u-element subsets of the ordered set $\hat{\boldsymbol{x}}_0 \in \binom{[\ell]}{j}_<$ in lexicographic order w.r.t. the indices. For every $\iota \in [j]$, we set

$$\partial_\iota \hat{\boldsymbol{x}}_u = \left(x_\Upsilon \colon \Upsilon \in \binom{\hat{\boldsymbol{x}}_0 \setminus \iota}{u}_< \right). \tag{7}$$

In other words, vector $\partial_\iota \hat{\boldsymbol{x}}_u$ contains precisely those entries of $\hat{\boldsymbol{x}}_u$ which are labeled by the u-element subsets of $\hat{\boldsymbol{x}}_0$ not containing ι. Note that, in view of (5), $\partial_\iota \hat{\boldsymbol{x}}_u = \hat{\boldsymbol{x}}_u^\Xi$ with $\Xi = \hat{\boldsymbol{x}}_0 \setminus \iota$. Clearly, $\partial_\iota \hat{\boldsymbol{x}}_u$ has $\binom{j-1}{u}$ entries from $[a_u]$. Furthermore, we set

$$\partial_\iota \hat{\boldsymbol{x}}^{(j-1)} = (\partial_\iota \hat{\boldsymbol{x}}_1, \partial_\iota \hat{\boldsymbol{x}}_2, \dots, \partial_\iota \hat{\boldsymbol{x}}_{j-1})$$

and observe that $\partial_\iota \hat{\boldsymbol{x}}^{(j-1)}$ is a $(j-1+2^{j-1}-1)$-dimensional vector belonging to $A(j-1, \boldsymbol{a})$.

3.2.3. *Regular Partitions and the Regularity Lemma.* The regularity lemma of Rödl and Skokan provides a family of partitions $\mathscr{R}$ with nice certain properties. Loosely speaking, for "most" $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$ the $(n/a_1, k, k-1)$-complex $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(k-1)})$ is regular (cf. Definition 8).

In the two definitions, below we introduce two concepts central to regularity. We use the notation δ', d' and r' to be consistent with the context in which we apply the Regularity Lemma (Theorem 26 below).

Definition 24 (**$(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable**)**.** *Let μ be a number in the interval $(0,1]$, let $\boldsymbol{\delta'} = (\delta'_2, \dots, \delta'_k)$ and $\boldsymbol{d'} = (d'_2, \dots, d'_k)$ be two arbitrary but fixed vectors of real numbers between 0 and 1 and let r' be a positive integer. We say that a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ (as defined in Definition 20) is $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable if all but μn^k edges of $K_\ell^{(k)}(V_1, \dots, V_\ell)$ belong to $(\boldsymbol{\delta'}, \boldsymbol{d'}, r')$-regular complexes $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(k-1)}) = \left\{ \hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(k-1)}) \right\}_{j=1}^{k-1}$ where $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$.*

Before finally stating the regularity lemma, we define regular partitions.

Definition 25 (**regular partition**)**.** *Let $\mathcal{G}^{(k)}$ be a (n, ℓ, k)-cylinder and let $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable family of partitions.*

We say $\mathscr{R}$ is a (δ'_k, r')-regular w.r.t. $\mathcal{G}^{(k)}$ if all but at most $\delta'_k n^k$ edges of $K_\ell^{(k)}(V_1, \dots, V_\ell)$ belong to polyads $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ such that $\mathcal{G}^{(k)}$ is (δ'_k, r')-regular with respect to $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

We now state the Regularity Lemma of Rödl and Skokan. In what follows, $\boldsymbol{D} = (D_2, \dots, D_{k-1})$ is a vector of positive real variables.

Theorem 26 (Regularity Lemma (ℓ-partite version))**.** *For all integers $\ell \ge k \ge 2$ and all positive reals δ'_k and μ and any positive functions*

$$\boldsymbol{\delta'}(\boldsymbol{D}) = \left(\delta'_{k-1}(D_{k-1}), \dots, \delta'_2(D_2, \dots, D_{k-1}) \right), \qquad r'(A_1, \boldsymbol{D}) = r'(A_1, D_2, \dots, D_{k-1}),$$

there exist integers n_k and L_k such that the following holds.

For any (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ with $n \ge n_k$ there exists a $\left(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r'(a_1, \boldsymbol{d'})\right)$-equitable $\left(\delta'_k, r'(a_1, \boldsymbol{d'})\right)$-regular family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ of density $\boldsymbol{d'} = \left(d'_2, \dots, d'_{k-1}\right)$ with $\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})| \le L_k$.

In the upcoming Corollary 28, we state an easy modification of Theorem 26 whose formulation is convenient for us in our proof of Theorem 9. Before stating Corollary 28, we outline the main differences between Theorem 26 and its corollary below. For that we need the following definition.

Definition 27 (**refinement**)**.** *Let $\{\mathcal{G}^{(h)}\}_{h=1}^j$ be an (n, ℓ, j)-complex and let $\mathscr{R} = \mathscr{R}(j, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(h)}\}_{h=1}^j$ be a family of partitions of $K_\ell^{(h)}(V_1, \dots, V_\ell)$ for $h \in [j]$. We say $\mathscr{R}$ refines $\{\mathcal{G}^{(h)}\}_{h=1}^j$ if for every $h \in [j]$ and every $\boldsymbol{x}^{(h)} \in A(h, \boldsymbol{a})$ either $\mathcal{R}^{(h)}(\boldsymbol{x}^{(h)}) \subseteq \mathcal{G}^{(h)}$ or $\mathcal{R}^{(h)}(\boldsymbol{x}^{(h)}) \cap \mathcal{G}^{(h)} = \varnothing$.*

Moreover, adding an additional layer $\mathcal{G}^{(j+1)} \subseteq \mathcal{K}_{j+1}^{(j)}(\mathcal{G}^{(j)})$ to $\{\mathcal{G}^{(h)}\}_{h=1}^j$, we will also say that $\mathscr{R} = \{\mathscr{R}^{(h)}\}_{h=1}^j$ refines the $(n, \ell, j+1)$-complex $\{\mathcal{G}^{(h)}\}_{h=1}^j \cup \mathcal{G}^{(j+1)}$ if $\mathscr{R}$ refines $\{\mathcal{G}^{(h)}\}_{h=1}^j$.

It is a well known fact that the proof of Szemerédi's regularity lemma not only yields the existence of a regular partition for any graph $G = \mathcal{G}^{(2)}$, but also shows that **any** given intitial partition $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ of the vertex set $V = V\left(\mathcal{G}^{(2)}\right)$ has a regular refinement. Similarly, the proof of Theorem 26 (which is proved by induction on k) yields immediately the existence of a regular and equitable partition $\mathscr{R}$ which refines a given partition of the underlying structure. In particular, regularizing the k-th layer $\mathcal{G}^{(k)}$ of a given (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$, one can obtain a partition $\mathscr{R} = \mathscr{R}(k-1,\boldsymbol{a},\boldsymbol{\varphi})$ satisfying the following property: *for any* $1 \le j \le k-1$ *and every* $\boldsymbol{x}^{(j)} \in A(j,\boldsymbol{a})$, *either* $\mathcal{R}^{(j)}\left(\boldsymbol{x}^{(j)}\right) \subseteq \mathcal{G}^{(j)}$ *or* $\mathcal{R}^{(j)}\left(\boldsymbol{x}^{(j)}\right) \cap \mathcal{G}^{(j)} = \varnothing$. In other words, $\mathscr{R}$ refines the partitions given by $\mathcal{G}^{(j)} \cup \overline{\mathcal{G}^{(j)}} = K^{(j)}(V_1,\dots,V_\ell)$ for every $j \in [k-1]$.

One can maintain yet another property of the $(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r')$-equitable family of partitions $\mathscr{R}$ with density vector $\boldsymbol{d'}$. In the proof of Theorem 26 (cf. [40]), the $\boldsymbol{d'}$ are chosen explicitly and there is a large freedom to choose them (more precisely there is no necessary lower bound on each d'_j, $2 \le j \le k-1$). Hence, we shall assume, without loss of generality, that for any given *fixed* $\sigma_2,\dots,\sigma_{k-1}$, we may arrange the constants d'_j, $2 \le j \le k-1$, so that the quotients σ_j/d'_j, $2 \le j \le k-1$, are integers.

Summarizing the discussion above we arrive at the following Corollary 28 (stated below) of Theorem 26. The full proof of Corollary 28 is identical to the proof of Theorem 26 with the two minor adjustments indicated above (see [40, Corollary 13.1]).

Corollary 28. *For all integers* $\ell \ge k \ge 2$ *and all positive reals* $\sigma_2,\dots,\sigma_{k-1}$, δ'_k, *and* μ *and all positive functions*

$$\boldsymbol{\delta'}(\boldsymbol{D}) = \left(\delta'_{k-1}(D_{k-1}),\dots,\delta'_2(D_2,\dots,D_{k-1})\right), \qquad r'(A_1,\boldsymbol{D}) = r'(A_1,D_2,\dots,D_{k-1}),$$

there exist integers n_k *and* L_k *such that the following holds.*

For every (n,ℓ,k)*-complex* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ *satisfying* $n \ge n_k$ *there exists a* $\left(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r'(a_1,\boldsymbol{d'})\right)$*-equitable* $\left(\delta'_k, r'(a_1,\boldsymbol{d'})\right)$*-regular (w.r.t.* $\mathcal{G}^{(k)}$*) family of partitions* $\mathscr{R} = \mathscr{R}(k-1,\boldsymbol{a},\boldsymbol{\varphi})$, $\boldsymbol{d'} = (d'_2,\dots,d'_{k-1})$, *such that*

- (i) $\mathscr{R}$ *refines* $\boldsymbol{\mathcal{G}}$,
- (ii) σ_j/d'_j *is an integer for* $j = 2,\dots,k-1$, *and*
- (iii) $\operatorname{rank}\mathscr{R} = |A(k-1,\boldsymbol{a})| \le L_k$.

3.2.4. *Statement of Cleaning Phase I.* The proof of the main theorem, Theorem 9, presented in Section 4 uses the following lemma, Lemma 30, which follows from Corollary 28 and the induction assumption on Theorem 9.

We use Lemma 30 in the proof of Theorem 9 instead of Corollary 28 since it allows a simpler presentation of the later arguments. For $k=2$, Lemma 30 is a straightforward reformulation of Szemerédi's theorem and reduces to the statement that for any graph $\mathcal{G}^{(2)} = (V,E)$, there is a graph $\tilde{\mathcal{G}}^{(2)}$ for which $\left|\mathcal{G}^{(2)} \triangle \tilde{\mathcal{G}}^{(2)}\right|$ is small and where $\tilde{\mathcal{G}}^{(2)} = (V,\tilde{E})$ admits a "perfectly equitable" partition, i.e., $V = V_1 \cup \dots \cup V_t$ with $|V_1| = \dots = |V_t|$ and all pairs (V_i,V_j) are ε-regular for $1 \le i < j \le t$. Lemma 30 will generalize this concept for $\mathcal{G}^{(k)}$ with $k > 2$.

The following definition reflects the "almost" ideal situation when, for each $2 \le j < k$, there is just one uncontrolable (but very small) "garbage partition class". Similarly to Definition 24 and Definition 25, we use tilde-notation in the next definition to be consistent with the context in which it is used.

Definition 29 (**almost perfect** $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$**-family**)**.** *Let* $\tilde{\boldsymbol{\delta}} = \left(\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}\right)$ *and* $\tilde{\boldsymbol{d}} = \left(\tilde{d}_2,\dots,\tilde{d}_{k-1}\right)$ *be vectors of reals,* $\tilde{r} > 0$ *be an integer and* $\boldsymbol{b} = (b_1,\dots,b_{k-1})$ *be a vector of positive integers. Set*

$$\bar{\boldsymbol{b}} = (b_1, b_2+1, b_3, \dots, b_{k-1}).$$

We say that a family of partitions $\mathscr{P} = \mathscr{P}(k-1,\bar{\boldsymbol{b}},\boldsymbol{\psi})$ *(as defined in Definition 20) is an almost perfect* $\left(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b}\right)$*-family of partitions if the following holds:*

- (i) $\tilde{d}_j b_j = 1 \pm \tilde{\delta}_j/\tilde{d}_j$ *for every* $2 \le j \le k-1$, *and*
- (ii) *for every* $2 \le j \le k-1$, *for every* $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1,\boldsymbol{b})$ *and for every* $\beta \in [b_j]$, *the* $(n/b_1,j,j)$*-cylinder* $\mathcal{P}^{(j)}\left((\hat{\boldsymbol{x}}^{(j-1)},\beta)\right)$ *is* $\left(\tilde{\delta}_j, \tilde{d}_j, \tilde{r}\right)$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}\left(\hat{\boldsymbol{x}}^{(j-1)}\right)$.

An almost perfect family of partitions $\mathscr{P}$ has the property that for every $\boldsymbol{x}^{(k-1)} \in A(k-1,\boldsymbol{b})$ the $(n/b_1,k-1,k-1)$-complex $\boldsymbol{\mathcal{P}}^{(k-1)}(\boldsymbol{x}^{(k-1)}) = \{\mathcal{P}^{(j)}(\boldsymbol{x}^{(k-1)})\}_{j=1}^{k-1}$ (cf. Claim 18) is 'garbage free', i.e., $\left((\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\dots,\tilde{d}_{k-1}),\tilde{r}\right)$-regular. In other words, all addresses $\boldsymbol{x}^{(k-1)}$ which give rise to irregular parts of the partition $\mathscr{P}$ are in $A(k-1,\bar{\boldsymbol{b}}) \setminus A(k-1,\boldsymbol{b})$.

The statement of the next lemma is very much tailored for its application in Section 4.4. While the constants $\delta_3, \dots, \delta_k$ are inessential for the proof of the following lemma, they allow a more convenient notation.

Lemma 30 (Cleaning Phase I)**.** *For every vector $\boldsymbol{d} = (d_2, \dots, d_k)$ of positive reals, for every choice of $\delta_3, \dots, \delta_k$, for any positive real $\tilde{\delta}_k$ and all positive functions*

$$\tilde{\boldsymbol{\delta}}(\boldsymbol{D}) = \Big(\tilde{\delta}_{k-1}\big(D_{k-1}\big), \dots, \tilde{\delta}_2\big(D_2, \dots, D_{k-1}\big)\Big)\,, \qquad \tilde{r}\big(B_1, \boldsymbol{D}\big) = \tilde{r}\big(B_1, D_2, \dots, D_{k-1}\big)\,,$$

there exist integers $\tilde{n}_k$, $\tilde{L}_k$, a vector of positive reals $\tilde{\boldsymbol{c}} = (\tilde{c}_2, \dots, \tilde{c}_{k-1})$ and a positive constant δ_2 so that the following holds:

For every $(\boldsymbol{\delta} = (\delta_2, \dots, \delta_k), \boldsymbol{d}, 1)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ with $n \geq \tilde{n}_k$ there exist an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, a vector $\tilde{\boldsymbol{d}} = \big(\tilde{d}_2, \dots, \tilde{d}_{k-1}\big)$ componentwise bigger than $\tilde{\boldsymbol{c}}$, and an almost perfect $\big(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(b_1, \tilde{\boldsymbol{d}}), \boldsymbol{b}\big)$-family of partitions $\mathscr{P} = \mathscr{P}\left(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi}\right) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ refining $\tilde{\boldsymbol{\mathcal{G}}}$ so that:

- (*i*) *$\tilde{\mathcal{G}}^{(k)}$ is $\big(\tilde{\delta}_k, \tilde{r}(b_1, \tilde{\boldsymbol{d}})\big)$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\left(k-1, \boldsymbol{b}\right)$,*
- (*ii*) *$\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$, $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$, and $\tilde{\mathcal{G}}^{(j)} \subseteq \mathcal{G}^{(j)}$ for every $3 \leq j \leq k$,*
- (*iii*) *for every $3 \leq j \leq k$ and every $j \leq i \leq \ell$, the following holds:*

$$\big|\mathcal{K}_i^{(j)}(\mathcal{G}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\big| = \big|\mathcal{K}_i^{(j)}(\mathcal{G}^{(j)}) \setminus \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\big| \leq \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,,$$

- (*iv*) *for every $\hat{\boldsymbol{x}}^{(1)} = ((\lambda_1, \lambda_2), (\beta_1, \beta_2)) \in \hat{A}(1, \boldsymbol{b})$, the graph $\tilde{\mathcal{G}}^{(2)}[V_{\lambda_1,\beta_1}, V_{\lambda_2,\beta_2}]$ is $(\tilde{L}_k^2\delta_2, d_2, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)}) = V_{\lambda_1,\beta_1} \cup V_{\lambda_2,\beta_2}$, and*
- (*v*) $\operatorname{rank} \mathscr{P} \leq \tilde{L}_k$ *and consequently $|\hat{A}(k-1, \boldsymbol{b})| \leq (\tilde{L}_k)^k$.*

In the proof of Lemma 30 (see Section 5) we construct an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}}$ admitting an almost perfect family of partitions $\mathscr{P}$. Moreover, $\tilde{\boldsymbol{\mathcal{G}}}$ is almost identical to a given (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ (see (*ii*) and (*iii*) of Lemma 30). In particular, $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$, while we allow a small difference between $\tilde{\mathcal{G}}^{(j)}$ and $\mathcal{G}^{(j)}$ for $j \geq 3$. For the special rôle of $\tilde{\mathcal{G}}^{(2)}$ we have to allow "garbage classes" in $\mathscr{P}^{(2)}$. These "garbage classes" are contained in $\psi_2^{-1}(b_2 + 1)$. Note that the integer vector $\bar{\boldsymbol{b}}$ differs from $\boldsymbol{b}$ only in the second coordinate.

On the other hand, note that the partition $\mathscr{P}$ given by Lemma 30 is perfect in the sense that for $2 \leq j \leq k$ every j-tuple of $\tilde{\mathcal{G}}^{(j)}$ belongs to a regular polyad of $\mathscr{P}$. This feature will later give us a significant notational advantage. On the other hand, Lemma 30 (*iii*) ensures that the two complexes $\boldsymbol{\mathcal{G}}$ and $\tilde{\boldsymbol{\mathcal{G}}}$ differ by few cliques only.

3.2.5. *The Slicing Lemma.* The following lemma whose proof is based on the fact that *randomly chosen subcylinders of a regular cylinder are regular* was proved in [40]. We will find it useful in this paper as well.

Lemma 31 (Slicing Lemma)**.** *Suppose ϱ, δ are two real numbers such that $0 < \delta/2 < \varrho \leq 1$. There is an $m_0 = m_0\left(\varrho, \delta\right)$ such that the following holds. If $\hat{\mathcal{P}}^{(j-1)}$ is an $(m, j, j-1)$-cylinder so that $\big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\big| \geq m^j/\ln m$ and if $\mathcal{F}^{(j)} \subseteq \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)$ is an (m, j, j)-cylinder which is $(\delta, \varrho, r_{\mathrm{SL}})$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$, then for every $0 < p < 1$, where $3\delta < p\varrho$ and $u = \lfloor 1/p \rfloor$, there exists a decomposition of $\mathcal{F}^{(j)} = \mathcal{F}_0^{(j)} \cup \mathcal{F}_1^{(j)} \cup \dots \cup \mathcal{F}_u^{(j)}$ such that $\mathcal{F}_i^{(j)}$ is $(3\delta, p\varrho, r_{\mathrm{SL}})$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ for $1 \leq i \leq u$.*

Moreover if $1/p$ is an integer then $\mathcal{F}_0^{(j)} = \varnothing$.

4. Proof of the Counting Lemma

The proof of the Counting Lemma, Theorem 9, is based on upcoming Theorem 34 stated in the Section 4.2. In Section 4.2, we prove Theorem 9 follows from Theorem 34. The remainder of the paper is devoted to the proof of Theorem 34, an outline of which is given in Section 4.3.

The proof of Theorem 34 splits into four lemmas, Lemmas 30, 37, 38 and 40. In Section 4.4, we show how Theorem 34 follows from these four lemmas. We defer the proofs of these lemmas to Sections 5–7.

The structure of the proof of the Counting Lemma outlined here will be summarized in Figure 1. Some further consequences of Theorem 34 are discussed in Section 8.

4.1. **Induction assumption on the Counting Lemma.** We prove the Counting Lemma, Theorem 9, by induction on k. For $k = 2$, the Counting Lemma, i.e, Fact 2, is a well known fact (see, e.g., [24, 25]) and for $k = 3$ it was proved by Nagle and Rödl in [28]. Therefore, from now on let $k \geq 4$ be a fixed integer. (For the inductive proof presented here it suffices to assume Fact 2 as the base case.)

Induction Hypothesis. *We assume that*

$$\mathbf{CL}_{j,i} \ \textit{holds for } 2 \leq j \leq k-1 \text{ and } i \geq j\,. \tag{8}$$

We prove $\mathbf{CL}_{k,\ell}$ holds for all integers $\ell \geq k$. For that, throughout the proof, let $\ell \geq k$ be fixed.

Rather than quoting various forms of our induction hypothesis $\mathbf{CL}_{j,i}$ (for varying $2 \leq j \leq k-1$ and $j \leq i \leq \ell$) involving different δ's and γ's, we summarize all such statements in one. The following statement, which we denote by $\mathbf{IHC}_{k-1,\ell}$, is a reformulation of our Induction Hypothesis.

Statement 32 (Induction Hypothesis on Counting)**.** *The following is true:* $\forall \eta > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \forall d_{k-2} > 0\ \exists \delta_{k-2} > 0\ \ldots\ \forall d_2 > 0\ \exists \delta_2 > 0$ *and there exist integers* r *and* $m_{k-1,\ell}$ *so that for all integers* j *and* i *with* $2 \leq j \leq k-1$ *and* $j \leq i \leq \ell$ *the following holds.*

If $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ *is a* $\big((\delta_2,\ldots,\delta_j),(d_2,\ldots,d_j),r\big)$*-regular* (m,i,j)*-complex with* $m \geq m_{k-1,\ell}$*, then*

$$\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}\big)\right| = (1 \pm \eta) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times m^i\,.$$

The following fact confirms that Statement 32 is an easy consequence of our Induction Hypothesis in (8).

Fact 33. *If* $\mathbf{CL}_{j,i}$ *holds for all integers* $2 \leq j \leq k-1$ *and* $j \leq i \leq \ell$*, then* $\mathbf{IHC}_{k-1,\ell}$ *holds.*

Note that Fact 33 is trivial to prove and only requires confirming the constants may be chosen appropriately; when given $\eta, d_{k-1}, \delta_{k-1}, \ldots, \delta_{j+1}$ and d_j, choose δ_j to be the minimum of all δ_j's from the statements $\mathbf{CL}_{h,i}$ with $j \leq h \leq k-1$ and $h \leq i \leq \ell$, where δ_j appears. Similarly, we set r and $m_{k-1,\ell}$ to the maximum of the corresponding constants in $\mathbf{CL}_{h,i}$.

As a consequence of the induction assumption stated in (8) and Fact 33, we may assume for the remainder of this paper that

$$\mathbf{IHC}_{k-1,\ell} \ \text{holds}\,. \tag{9}$$

4.2. **Proof of Theorem 9.** The proof of the Counting Lemma, Theorem 9, consists of two main parts. The first part is Theorem 34, stated below, which receives input an (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ from the hypothesis of the Counting Lemma, Theorem 9. Theorem 34 then guarantees the existence of an (n,ℓ,k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(h)}\}_{h=1}^{k}$ having the following properties.

- (a) The complex $\boldsymbol{\mathcal{F}}$ differs only slightly from $\boldsymbol{\mathcal{G}}$. In particular, the number of ℓ-cliques in $\mathcal{G}^{(k)}$ and $\mathcal{F}^{(k)}$ are essentially the same (see property (iii) of Theorem 34).
- (b) The complex $\boldsymbol{\mathcal{F}}$ is "ready" for an application of the Dense Counting Lemma, Theorem 16.

The proof of the following Theorem 34 is based on the induction hypothesis, $\mathbf{IHC}_{k-1,\ell}$ (cf. Statement 32). In the formulation below, the integer k is already fixed (cf. Section 4.1) according to our induction hypothesis.

Theorem 34. *Let* $\ell \geq k$ *be a fixed integer. The following is true:* $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \ldots\ \forall d_2 > 0, \varepsilon > 0\ \exists \delta_2 > 0$ *and there exist integers* r *and* n_0 *so that, with* $\boldsymbol{d} = (d_2,\ldots,d_k)$ *and* $\boldsymbol{\delta} = (\delta_2,\ldots,\delta_k)$ *and* $n \geq n_0$*, whenever* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ *is a* $(\boldsymbol{\delta},\boldsymbol{d},r)$*-regular* (n,ℓ,k)*-complex, then there exists an* (n,ℓ,k)*-complex* $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(h)}\}_{h=1}^{k}$ *such that*

- (i) $\boldsymbol{\mathcal{F}}$ *is* $(\boldsymbol{\varepsilon},\boldsymbol{d},1)$*-regular, with* $\boldsymbol{\varepsilon} = (\varepsilon,\ldots,\varepsilon) \in \mathbb{R}^{(k-1)}$*,*
- (ii) $\mathcal{F}^{(1)} = \mathcal{G}^{(1)}$ *and* $\mathcal{F}^{(2)} = \mathcal{G}^{(2)}$*, and*
- (iii) $\big|\mathcal{K}_\ell^{(k)}\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{F}^{(k)}\big)\big| \leq (\gamma/2) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell$.

We mention that Theorem 34 has some interesting implications of its own which we discuss in Section 8.

We present a proof of Theorem 34 in Section 4.4. In the immediate sequel, we give the proof of the Inductive Step for the Counting Lemma based on Theorem 34 and Theorem 16. We note that this proof of $\mathbf{CL}_{k,\ell}$ does not directly use the induction hypothesis $\mathbf{IHC}_{k-1,\ell}$, but we will use $\mathbf{IHC}_{k-1,\ell}$ in the proof of Theorem 34 (see Figure 1).

Proof of the Induction Step. We begin by describing the constants involved. With the exception of ε, Theorem 9 and Theorem 34 involve the same constants under the same quantification. Hence, given γ and d_k from Theorem 9, we let δ_k be the $\delta_k\big(\text{Thm.34}(\gamma, d_k)\big)$ from Theorem 34. In general, given d_j, $j = k, \ldots, 3$, we set

$$\delta_j = \delta_j\big(\text{Thm.34}(\gamma, d_k, \delta_k, d_{k-1}, \ldots, \delta_{j+1}, d_j)\big)\,.$$

Having fixed $\gamma, d_k, \delta_k, d_{k-1}, \ldots, \delta_4, d_3, \delta_3$, now let d_2 be given by Theorem 9. Next, we fix ε for Theorem 34 so that

$$\varepsilon \le \varepsilon\big(\text{Thm.16}(d_2, \ldots, d_k)\big) \quad \text{and} \quad g_{k,\ell}(\varepsilon) \le \frac{\gamma}{2}\,, \tag{10}$$

where $g_{k,\ell}$ is given by the Dense Counting Lemma, Theorem 16. Moreover, let $m_0\big(\text{Thm.16}(d_2, \ldots, d_k)\big)$ be the lower bound on the number of vertices given by Theorem 16 applied to $d_2, \ldots, d_k$.

Then, Theorem 34 yields

$$\begin{gathered}\delta_2\big(\text{Thm.34}(\gamma, d_k, \delta_k, \ldots, \delta_3, d_2, \varepsilon)\big)\,, \quad r\big(\text{Thm.34}(\gamma, d_k, \delta_k, \ldots, \delta_3, d_2, \varepsilon)\big)\,, \\ \text{and} \quad n_0\big(\text{Thm.34}(\gamma, d_k, \delta_k, \ldots, \delta_3, d_2, \varepsilon)\big)\,.\end{gathered} \tag{11}$$

Finally, we set δ_2 and r for Theorem 9 to its corresponding constants given in (11). Also, we set n_0 for Theorem 9 to

$$n_0 = \max\big\{n_0\big(\text{Thm.34}(\gamma, d_k, \delta_k, \ldots, \delta_3, d_2, \varepsilon)\big), m_0\big(\text{Thm.16}(d_2, \ldots, d_k)\big)\big\}\,.$$

Now, let $\boldsymbol{\mathcal{G}}$ be a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex satisfying $n \ge n_0$. Then, Theorem 34 yields an $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}}$ satisfying (i)–(iii) of Theorem 34. Consequently, by (10) and (i), we may apply the Dense Counting Lemma to $\boldsymbol{\mathcal{F}}$. Therefore,

$$\Big|\mathcal{K}_\ell^{(k)}\big(\mathcal{F}^{(k)}\big)\Big| = \Big(1 \pm \frac{\gamma}{2}\Big)\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell$$

and Theorem 9 follows from (iii) of Theorem 34. □

We note that the proof of Theorem 9 did not use the full strength of Theorem 34. In particular, we made no use of (ii) here. However, (ii) is important with respect to further consequences of Theorem 34 discussed in Section 8.

4.3. **Outline of the proof of Theorem 34.** Given an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$, Theorem 34 ensures the existence of an appropriate (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$. This complex is constructed successively in three phases outlined below.

The first phase we call Cleaning Phase I and is a variant of the RS-Lemma (see Theorem 26). The lemma corresponding to Cleaning Phase I, Lemma 30, was already stated in Section 3.2.4. Given a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular input complex $\boldsymbol{\mathcal{G}}$ with $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$ and $\boldsymbol{d} = (d_2, \ldots, d_k)$, we fix

$$\tilde{\delta}_k \ll \varepsilon' \ll \min\{\varepsilon, d_2, \ldots, d_k\}. \tag{12}$$

Lemma 30 alters $\boldsymbol{\mathcal{G}}$ slightly (this is measured by $\tilde{\delta}_k$) to obtain an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$ together with an almost perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(\tilde{\boldsymbol{d}}), \boldsymbol{b})$-family of partitions which is $(\tilde{\delta}_k, \tilde{r}(\tilde{\boldsymbol{d}}))$-regular w.r.t. $\tilde{\mathcal{G}}^{(k)}$ (cf. Lemma 30 and Figure 2 in Section 4.4.2). Importantly, Lemma 30 (iii) will ensure that

$$\Big|\mathcal{K}_\ell^{(k)}\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big)\Big| \quad \text{is “small”}. \tag{13}$$

Cleaning Phase I enables us to work with a complex $\tilde{\boldsymbol{\mathcal{G}}}$ admiting a partition with almost no irregular polyads. These details are done largely for convenience to help ease subsequent parts of the proof.

We next proceed to Cleaning Phase II with the complex $\tilde{\boldsymbol{\mathcal{G}}}$ and an almost perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(\tilde{\boldsymbol{d}}), \boldsymbol{b})$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$, rank $\mathscr{P} \le \tilde{L}_k$ (cf. Lemma 30). Since $\tilde{\boldsymbol{\mathcal{G}}}$ differs from $\boldsymbol{\mathcal{G}}$ only slightly (this is measured by $\tilde{\delta}_k$), it follows from the choice of constants (argued in Fact 44) that $\tilde{\boldsymbol{\mathcal{G}}}$ inherits $(2\boldsymbol{\delta}, \boldsymbol{d}, r)$-regularity from $\boldsymbol{\mathcal{G}}$. Moreover, the choice of r ensuring $r \ge \tilde{L}_k$ (cf. (29)) implies that the density $d\big(\tilde{\mathcal{G}}^{(j)}|\hat{\mathcal{P}}^{(j-1)}\big)$ is close to what it “should be”, namely, d_j, for “most” polyads $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)} \subseteq \tilde{\mathcal{G}}^{(j-1)}$.

The goal in Cleaning Phase II is to perfect the small number of polyads having aberrant density. More specifically, Cleaning Phase II constructs “unidense” (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$ where $d\big(\mathcal{F}^{(j)}|\hat{\mathcal{P}}^{(j-1)}\big)$

is the same and equal to d_j for every polyad $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)} \subseteq \mathcal{F}^{(j-1)}$. The importance of "unidensity" is that it allows us to apply the Union Lemma, Lemma 40. Then the "final product" of the Union Lemma, the (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}}$, will satisfy (i) of Theorem 34. We now further examine the details of Cleaning Phase II.

Cleaning Phase II splits into two parts. In Part 1 of Cleaning Phase II (cf. Lemma 37), we correct the first $k-1$ layers of possible imperfections of $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, $j < k$, by constructing a "unidense" $(n, \ell, k-1)$-complex $\boldsymbol{\mathcal{H}^{(k-1)}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$. For the construction of $\boldsymbol{\mathcal{H}^{(k-1)}}$, we need to count cliques in such a complex and use the powerful tool $\mathbf{IHC}_{k-1,\ell}$ which we have available by the induction assumption ($\mathbf{CL}_{j,\ell}$ for $2 \leq j \leq k-1$).

We next remedy imperfections on the k-th layer $\tilde{\mathcal{G}}^{(k)}$. However, in the absence of our Induction Assumption herein, we have to proceed more carefully.

We first construct a still "somewhat imperfect" (n, ℓ, k)-cylinder $\mathcal{H}^{(k)}$ so that $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ is an (n, ℓ, k)-complex and $d\big(\mathcal{H}^{(k)} | \hat{\mathcal{P}}^{(k-1)}\big) = d_k \pm \sqrt{\delta_k}$ for every polyad $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$. While $\tilde{\mathcal{G}}^{(k)}$ satisfies that for "most" $\hat{\mathcal{P}}^{(k-1)} \subseteq \tilde{\mathcal{G}}^{(k-1)}$, its density is close to d_k, the new $\mathcal{H}^{(k)}$ has density close to d_k for "all" $\hat{\mathcal{P}}^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$. Moreover, we can construct $\mathcal{H}^{(k)}$ in such a way that

$$\left|\mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big)\right| \quad \text{is "small"}. \tag{14}$$

Part 2 of Cleaning Phase II deals with $\mathcal{H}^{(k)}$, the k-th layer of the complex $\boldsymbol{\mathcal{H}}$. In this part, we construct unidense (w.r.t. $\mathcal{H}^{(k-1)}$ and $\mathscr{P}^{(k-1)}$) (n, ℓ, k)-cylinders $\mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)}$ (with densities $d_k - \sqrt{\delta_k}$ and $d_k + \sqrt{\delta_k}$, respectively) and $\mathcal{F}^{(k)}$ where each of these cylinders, together with $\boldsymbol{\mathcal{H}^{(k-1)}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$, forms an (n, ℓ, k)-complex. In the construction, we will also ensure that

$$\mathcal{H}_-^{(k)} \subseteq \mathcal{H}^{(k)} \subseteq \mathcal{H}_+^{(k)} \qquad\qquad \mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}\,. \tag{15}$$

We then set $\mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $j < k$ and $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$.

We now discuss how we infer (i) and (iii) of Theorem 34 for $\boldsymbol{\mathcal{F}}$ (property (ii) is somewhat technical and we omit it from the outline given here). For part (i) of Theorem 34, we need to show that $\mathcal{F}^{(j)}$ is $(\varepsilon, d_j, 1)$-regular w.r.t. $\mathcal{F}^{(j-1)}$, $2 \leq j \leq k$. To this end, we take the union of all "partition blocks" from $\mathscr{P}^{(j)}$ (which are subhypergraphs of $\mathcal{F}^{(j)}$). Note that all these blocks are very regular (w.r.t. their underlying polyads (which are subhypergraphs of $\mathcal{F}^{(j-1)}$)) and have the same relative density (due to the unidensity). In fact, these blocks will be so regular that their union is $(\varepsilon', d_j, 1)$-regular and therefore also $(\varepsilon, d_j, 1)$-regular (cf. (12)). Consequently, as proved in the Union Lemma, Lemma 40, we obtain that $\boldsymbol{\mathcal{F}}$ is $\big((\varepsilon, \dots, \varepsilon), \boldsymbol{d}, 1\big)$-regular which proves (i) of Theorem 34.

We now outline the proof of part (iii). Observe that

$$\begin{aligned}\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{F}^{(k)}\big)\right| \leq \left|\mathcal{K}_\ell^{(k)}\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big)\right| &+ \\ + \left|\mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big)\right| &+ \\ + \left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{F}^{(k)}\big)\right|&.\end{aligned} \tag{16}$$

The first two terms of the right-hand side are small as mentioned earlier (see (13) and (14)). Let us say a few words on how to bound the third quantity.

Because of (15), we have

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{F}^{(k)}\big)\right| \leq \left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}_+^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{H}_-^{(k)}\big)\right| = \left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}_+^{(k)}\big)\right| - \left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}_-^{(k)}\big)\right|. \tag{17}$$

Since both complexes $\boldsymbol{\mathcal{H}_+}$ and $\boldsymbol{\mathcal{H}_-}$ are unidense, we can show that $\boldsymbol{\mathcal{H}_*}$ is $\big((\varepsilon', \dots, \varepsilon'), \boldsymbol{d_*}, 1\big)$-regular for $* \in \{+, -\}$ where $\boldsymbol{d_*} = (d_2, \dots, d_{k-1}, d_k^*)$ and $d_k^* = d_k + \sqrt{\delta_k}$ for $* = +$ and $d_k^* = d_k - \sqrt{\delta_k}$ for $* = -$. Similarly to the proof of (i), where the Union Lemma was applied to $\boldsymbol{\mathcal{F}}$, we can use it here for $\boldsymbol{\mathcal{H}_+}$ and $\boldsymbol{\mathcal{H}_-}$. Consequently, owing to (12), we can apply the Dense Counting Lemma, Theorem 16, to bound the right-hand side of (17) and thus the right-hand side of (16). This yields part (iii) of Theorem 34.

The flowchart in Figure 1 gives a sketch of the connection of theorems and lemmas involved in the proof of $\mathbf{CL}_{k,\ell}$, Theorem 9. Each box represents a theorem or lemma containing a reference for its proof. Vertical

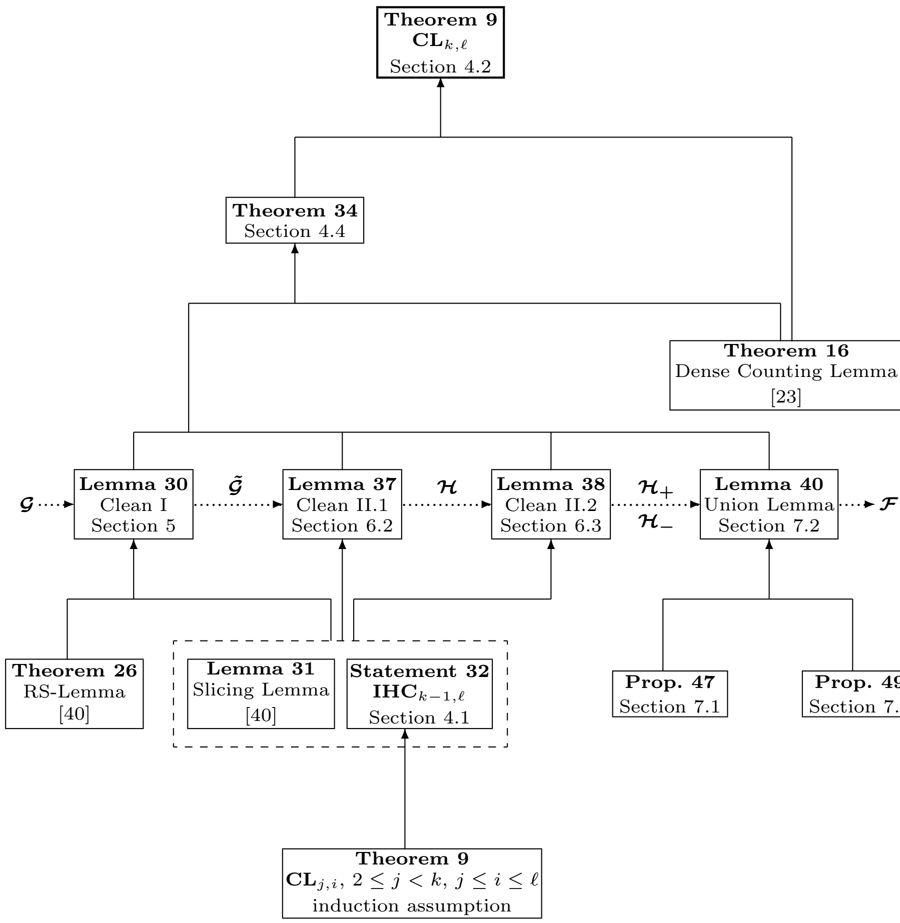

FIGURE 1. Structure of the proof of Theorem 9

arcs indicate which statements are needed to prove the statement to which the arc points. The horizontal arcs indicate the alteration of the involved complexes outlined above.

4.4. **Proof of Theorem 34.** In this section, we give all the details of the proof of Theorem 34 outlined in the last section. The proof of Theorem 34 splits into four parts. We separate these parts across Sections 4.4.1–4.4.4.

4.4.1. *Constants.* The hierarchy of the involved constants plays an important rôle in our proof. The choice of the constants breaks into two steps.

Step 1. Let an integer ℓ be given. We first recall the quantification of Theorem 34:

$$\forall \gamma, d_k \ \exists \delta_k \ \forall d_{k-1} \dots \exists \delta_3 \ \forall d_2, \varepsilon \ \exists \delta_2, r, n_0 \,.$$

Given γ and d_k we choose δ_k such that

$$\delta_k \lll \min\{\gamma, d_k\} \tag{18}$$

holds. Now, let d_{k-1} be given. We set

$$\eta = 1/4 \,. \tag{19}$$

(Our proof is not too sensitive to the choice of η, representing the multiplicative error for $\mathbf{IHC}_{k-1,\ell}$.) We then choose δ_{k-1} in such a way that $\delta_{k-1} \lll \min\{\delta_k, d_{k-1}\}$ and $\delta_{k-1} \le \delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1})\big)$ where $\delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1})\big)$ is the value of δ_{k-1} given by Statement 32 for η and d_{k-1}. We then proceed and define δ_j for $j = k-2, \dots, 3$, in the similar way. Summarizing the above, for $j = k-1, \dots, 3$ we choose δ_j such that

$$\delta_j \lll \min\{\delta_{j+1}, d_j\} \qquad \text{and} \qquad \delta_j \le \delta_j\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \dots, \delta_{j+1}, d_j)\big)\,. \tag{20}$$

We mention that after d_2 is revealed, we pause before defining δ_2.

Indeed, next we choose an auxiliary constant ε' so that

$$\begin{aligned} &\varepsilon' \le \min\Big\{\varepsilon\big(\text{Thm.16}(d_2, \dots, d_{k-1}, d_k - \delta_k^{1/2})\big), \varepsilon\big(\text{Thm.16}(d_2, \dots, d_{k-1}, d_k + \delta_k^{1/2})\big)\Big\}\,, \\ &\varepsilon' \lll \min\big\{\delta_3, d_2, \varepsilon\big\}\,, \qquad \text{and} \qquad g_{k,\ell}(\varepsilon') \lll \delta_k\,, \end{aligned} \tag{21}$$

where $g_{k,\ell}$ is given by the Dense Counting Lemma, Theorem 16. We then fix $\tilde\eta$, ν, and $\tilde\delta_k$ to satisfy

$$\varepsilon' \ggg \tilde\eta \ggg \nu \ggg \tilde\delta_k \qquad \text{and} \qquad \tilde\delta_k \le 1/8\,. \tag{22}$$

This completes Step 1 of the choice of the constants. We summarize the choices above in the following flowchart:

$$\begin{array}{ccccccccccccc} & & d_k & & \dots & & d_3 & & d_2, \varepsilon & & & & \\ & & \Downarrow & & \dots & & \Downarrow & & \Downarrow & & & & \\ \gamma & \ggg & \delta_k & \ggg & \cdots \ggg & & \delta_3 & \ggg & \varepsilon' & \ggg & \tilde\eta \ggg & \nu & \ggg \tilde\delta_k \end{array} \tag{23}$$

Step 2. The definition of the constants here is more subtle. Our goal is to extend (23) with the additional constants $\tilde d_j$, $\tilde\delta_j$ (for $j = k-1, \dots, 2$), $\tilde r$, $\tilde L_k^2\delta_2$ and r so that

$$\begin{array}{ccccccccc} & & \tilde d_{k-1} & & \dots & \tilde d_3 & & \tilde d_2 & \\ & & \Downarrow & & \dots & \Downarrow & & \Downarrow & \\ \tilde\delta_k & \ggg & \tilde\delta_{k-1} & \ggg & \cdots \ggg & \tilde\delta_3 & \ggg & \tilde\delta_2, 1/\tilde r & \ggg \quad \tilde L_k^2\delta_2, 1/r \end{array}$$

In our proof, we apply Lemma 30 to the (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$. Lemma 30 has functions $\tilde\delta_j(D_j, \dots, D_{k-1})$ for $j = 2, \dots, k-1$ and $\tilde r(D_2, \dots, D_{k-1})$ in variables $D_2, \dots, D_{k-1}$ as part of its input. The application of Lemma 30 results in an almost perfect $\big(\tilde{\boldsymbol\delta}(\tilde{\boldsymbol d}), \tilde{\boldsymbol d}, \tilde r(\tilde{\boldsymbol d}), \boldsymbol b\big)$-family of partitions $\mathscr{P}$ with some $\tilde{\boldsymbol d} = (\tilde d_2, \dots, \tilde d_{k-1})$. We want to be able to count cliques within the polyads of the family of regular partitions $\mathscr{P}$ by applying $\mathbf{IHC}_{k-1,\ell}$. Therefore, we choose the functions $\tilde\delta_j(D_j, \dots, D_{k-1})$ for Lemma 30 in such a way that they comply with the quantification of $\mathbf{IHC}_{k-1,\ell}$, Statement 32.

To this end let $\tilde\delta_{k-1}(D_{k-1})$ be a function satisfying $\tilde\delta_{k-1}(D_{k-1}) \lll \min\{\tilde\delta_k, D_{k-1}\}$, $\tilde\delta_{k-1}(D_{k-1}) \le \delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\tilde\eta, D_{k-1})\big)$ and $(1 + \tilde\delta_{k-1}(D_{k-1})/D_{k-1})^{\binom{\ell}{k-1}} < (1+\nu)^{1/k-2}$. We next choose the function $\tilde\delta_{k-2}(D_{k-2}, D_{k-1})$ in a similar way, making sure that

$$\begin{aligned} &\tilde\delta_{k-2}(D_{k-2}, D_{k-1}) \lll \min\big\{\tilde\delta_{k-1}(D_{k-1}), D_{k-2}\big\}\,, \\ &\tilde\delta_{k-2}(D_{k-2}, D_{k-1}) \le \delta_{k-2}\big(\mathbf{IHC}_{k-1,\ell}(\tilde\eta, D_{k-1}, \tilde\delta_{k-1}(D_{k-1}), D_{k-2})\big)\,, \end{aligned}$$

and

$$\left(1 + \frac{\tilde\delta_{k-2}(D_{k-2}, D_{k-1})}{D_{k-2}}\right)^{\binom{\ell}{k-2}} < (1+\nu)^{1/k-2}\,.$$

(Since $\tilde\delta_{k-1}(D_{k-1})$ is a function of D_{k-1} and $\tilde\eta$ was fixed in (22) already, we indeed also have that the right-hand sides of the first two inequalities above depend on the variables D_{k-2} and D_{k-1} only.) In general for $j = k-1, \ldots, 2$ we choose $\tilde{\delta}_j(D_j, \ldots, D_{k-1})$ so that

$$\begin{aligned}
&\tilde{\delta}_j(D_j, \ldots, D_{k-1}) \lll \min\left\{D_j, \tilde{\delta}_{j+1}(D_{j+1}, \ldots, D_{k-1})\right\}, \\
&\tilde{\delta}_j(D_j, \ldots, D_{k-1}) \leq \delta_j\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), D_{k-2}, \ldots, \tilde{\delta}_{j+1}(D_{j+1}, \ldots, D_{k-1}), D_j)\big) \\
&\text{and} \\
&\left(1 + \frac{\tilde{\delta}_j(D_j, \ldots, D_{k-1})}{D_j}\right)^{\binom{\ell}{j}} < (1+\nu)^{1/k-2}.
\end{aligned} \tag{24}$$

We may assume, without loss of generality, that the functions defined in (24) are componentwise monotone decreasing. Since for every $h \geq j+1$ the $\tilde{\delta}_h$ was constructed as a function of $D_h, \ldots, D_{k-1}$ only, as before, we may view the right-hand sides of the first two inequalities of (24) as a function of $D_j, \ldots, D_{k-1}$ only. Consequently, $\tilde{\delta}_j$ is a function of $D_j, \ldots, D_{k-1}$, as promised. Furthermore, we set $\tilde{r}(D_2, \ldots, D_{k-1})$ to be a componentwise monotone increasing function such that

$$\begin{aligned}
&\tilde{r}(D_2, \ldots, D_{k-1}) \ggg \max\left\{1/D_2, 1/\tilde{\delta}_3(D_3, \ldots, D_{k-1})\right\} \qquad \text{and} \\
&\tilde{r}(D_2, \ldots, D_{k-1}) \geq r\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), \ldots D_2)\big).
\end{aligned} \tag{25}$$

As a result of Lemma 30 applied to the constants $\boldsymbol{d} = (d_2, \ldots, d_k)$, $\delta_3, \ldots, \delta_k$, $\tilde{\delta}_k$, and the functions $\tilde{\delta}_{k-1}(D_{k-1}), \ldots, \tilde{\delta}_2(D_2, \ldots, D_{k-1})$ and $\tilde{r}(D_2, \ldots, D_{k-1})$ we obtain integers $\tilde{n}_k$, $\tilde{L}_k$, a vector of positive reals $\tilde{\boldsymbol{c}} = (\tilde{c}_2, \ldots, \tilde{c}_{k-1})$ and a constant $\delta_2^{\text{Lem.30}}$. (Here we did not use the variable B_1 for the function $\tilde{r}(D_2, \ldots, D_{k-1})$.) Next, we disclose δ_2 and r promised by Theorem 9. For that we apply the functions $\tilde{\delta}_2(D_2, \ldots, D_{k-1})$ and $\tilde{r}(D_2, \ldots, D_{k-1})$, defined in (24) and (25), to $\tilde{\boldsymbol{c}}$. We set δ_2 and r so that

$$\big(\tilde{L}_k^2 \delta_2\big) \lll \min\left\{\tilde{\delta}_2(\tilde{\boldsymbol{c}}), 1/\tilde{r}(\tilde{\boldsymbol{c}})\right\} \delta_2 \leq \delta_2\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \ldots, \delta_3, d_2)\big), \tag{26}$$

and

$$r \ggg \max\left\{1/\tilde{\delta}_2(\tilde{\boldsymbol{c}}), \tilde{r}(\tilde{\boldsymbol{c}}), 2^\ell \tilde{L}_k^k\right\} r \geq r\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \ldots, \delta_3, d_2)\big). \tag{27}$$

Finally, we set n_0 so that

$$n_0 \ggg \max\left\{\tilde{n}_k,\ 1/\delta_2,\ r,\ m_0\, \tilde{L}_k m_{k-1,\ell},\ \tilde{L}_k \tilde{m}_{k-1,\ell}\right\}, \tag{28}$$

where

$$m_0 = \max\left\{m_0\big(\text{Thm.16}(d_2, \ldots, d_k - \delta_k^{1/2})\big), m_0\big(\text{Thm.16}(d_2, \ldots, d_k + \delta_k^{1/2})\big)\right\}$$

is given by Theorem 16 applied to $d_2, \ldots, d_{k-1}$ and $d_k - \delta_k^{1/2}$ and $d_k + \delta_k^{1/2}$, respectively, and similarly

$$\begin{aligned}
&m_{k-1,\ell} = m_{k-1,\ell}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, \ldots, d_2, \delta_2)\big) \quad \text{and} \\
&\tilde{m}_{k-1,\ell} = m_{k-1,\ell}\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, \tilde{c}_{k-1}, \tilde{\delta}_{k-1}(\tilde{c}_{k-1}), \ldots, \tilde{c}_2, \tilde{\delta}_2(\tilde{\boldsymbol{c}}))\big)
\end{aligned}$$

come from Statement 32.

We now defined all constants involved in the statement Theorem 34. Moreover, we defined the functions and constants needed for Lemma 30. This brings us to the next part of the proof, Cleaning Phase I.

4.4.2. *Cleaning Phase I.* Let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ be a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex where $n \geq n_0$ and $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$, $\boldsymbol{d} = (d_2, \ldots, d_k)$ and r are chosen as described in Section 4.4.1. We apply Lemma 30 to $\boldsymbol{\mathcal{G}}$ with the constant $\tilde{\delta}_k$, the functions $\tilde{\boldsymbol{\delta}}(\boldsymbol{D}) = (\tilde{\delta}_{k-1}(D_{k-1}), \ldots, \tilde{\delta}_2(D_2, \ldots, D_{k-1}))$ and the function $\tilde{r}(\boldsymbol{D})$ as given in (22), (24) and (25). Lemma 30 renders an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$, a real vector of positive coordinates $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \ldots, \tilde{d}_{k-1})$ componentwise bigger than $\tilde{\boldsymbol{c}}$ and an almost perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ refining $\tilde{\boldsymbol{\mathcal{G}}}$ (cf. Definition 27 and Definition 29). Note that the choice of r in (27) and (v) of Lemma 30 ensures that for $2 \leq j \leq k$,

$$r \geq 2^\ell \tilde{L}_k^k \geq 2^\ell \big|\hat{A}(k-1, \boldsymbol{b})\big| \geq \big|\hat{A}(j-1, \boldsymbol{b})\big|. \tag{29}$$

For $2 \leq j \leq k-1$, we finally fix the constants

$$\tilde{\delta}_j = \tilde{\delta}_j(\tilde{d}_j, \ldots, \tilde{d}_{k-1}) \qquad \text{and} \qquad \tilde{r} = \tilde{r}(\tilde{\boldsymbol{d}}).$$

From the monotonicity of the functions $\tilde{\delta}_2$ and $\tilde{r}$, we infer

$$\tilde{\delta}_2 = \tilde{\delta}_2(\tilde{\boldsymbol{d}}) \geq \tilde{\delta}_2(\tilde{\boldsymbol{c}}) \ggg \delta_2 \quad \text{and} \quad \tilde{r} = \tilde{r}(\tilde{\boldsymbol{d}}) \leq \tilde{r}(\tilde{\boldsymbol{c}}) \lll r\,. \tag{30}$$

For future reference, we summarize, in Figure 2, (18)–(27) and (30).

$$\begin{array}{ccccccccccccc}
 & & d_k & & \dots & d_3 & & d_2, \varepsilon & & & & & \\
 & & \rotatebox{-90}{$\ggg$} & & \dots & \rotatebox{-90}{$\ggg$} & & \rotatebox{-90}{$\ggg$} & & & & & \\
\gamma & \ggg & \delta_k & \ggg \cdots \ggg & & \delta_3 & \ggg & \varepsilon' & & & & & \\
 & & & & & & & \rotatebox{-90}{$\ggg$} & & & & & \\
 & & & & & & & \tilde{\eta} & & & & & \\
 & & & & & & & \rotatebox{-90}{$\ggg$} & & & & & \\
 & & & & & & & \nu & & \tilde{d}_{k-1} & \dots & \tilde{d}_3 & \tilde{d}_2 \\
 & & & & & & & \rotatebox{-90}{$\ggg$} & & \rotatebox{-90}{$\ggg$} & \dots & \rotatebox{-90}{$\ggg$} & \rotatebox{-90}{$\ggg$} \\
 & & & & & & & \tilde{\delta}_k & \ggg & \tilde{\delta}_{k-1} & \ggg \cdots \ggg & \tilde{\delta}_3 \ \ggg & \tilde{\delta}_2, \frac{1}{\tilde{r}} \\
 & & & & & & & & & & & & \rotatebox{-90}{$\ggg$} \\
 & & & & & & & & & & & & \tilde{L}_k^2 \delta_2, \frac{1}{r}
\end{array}$$

FIGURE 2. Flowchart of the constants

For the remainder of this paper, all constants are fixed as summarized above in Figure 2.

Observe that by the choice of the functions $\tilde{\delta}_j$ in (24) and part (i) of Definition 29, we have for every $2 \leq j < k$ and $j < i \leq \ell$

$$\prod_{h=2}^{j} (\tilde{d}_h b_h)^{\binom{i}{h}} = \prod_{h=2}^{j} \left(1 \pm \tilde{\delta}_h / \tilde{d}_h\right)^{\binom{i}{h}} = \prod_{h=2}^{j} (1 \pm \nu)^{1/k-2} = 1 \pm \nu\,. \tag{31}$$

Remark 35. *Observe that the last two "equality signs" in* (31) *are used in a non-symmetric way. For example, the last equality sign abbreviates the validity of the two inequalities*

$$(1-\nu) \leq \prod_{h=2}^{j} (1-\nu)^{1/k-2} \qquad \textit{and} \qquad \prod_{h=2}^{j} (1+\nu)^{1/k-2} \leq (1+\nu)\,.$$

We will use this notation occasionally in the calculations throughout this paper.

Part (iii) of Lemma 30 bounds the difference of the number of $K_\ell^{(k)}$'s in $\mathcal{G}^{(k)}$ and $\tilde{\mathcal{G}}^{(k)}$ by

$$\tilde{\delta}_k \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell \lll \delta_k \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{32}$$

For future reference, we summarize the results of Cleaning Phase I.

Setup 36 (After Cleaning Phase I)**.** *Let all constants be chosen as summarized in Figure 2 so that also* (29) *and* (31) *hold. Let $\boldsymbol{\mathcal{G}}$ be the $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex from the input of the Counting Lemma, Theorem 9. Let $\tilde{\boldsymbol{\mathcal{G}}}$ be the (n, ℓ, k)-complex and $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \psi)$ be the almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions refining $\tilde{\boldsymbol{\mathcal{G}}}$ given after Cleaning Phase I, i.e., after an application of Lemma 30.*

We now mention a few comments to motivate our next step in the proof. The family of partitions $\mathscr{P}$ given by Lemma 30 (cf. Setup 36) is an almost perfect family (cf. Definition 29); moreover, by (i), $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$. However, while every component of the partition is regular, it is possible that the densities $d\big(\tilde{\mathcal{G}}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$ may vary across different $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b})$ for which $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq \tilde{\mathcal{G}}^{(j-1)}$.

The goal of the next cleaning phase is to alter $\tilde{\boldsymbol{\mathcal{G}}}$ to form a complex $\boldsymbol{\mathcal{F}}$ where all densities are appropriately uniform. Importantly, we show that the two complexes $\tilde{\boldsymbol{\mathcal{G}}}$ and $\boldsymbol{\mathcal{F}}$ share mostly all their respective cliques. (For technical reasons, we will also need to construct two auxiliary complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$)

4.4.3. *Cleaning Phase II.* The aim of this section is to construct the complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$ which is promised by Theorem 34. For the proof of part (*iii*) of Theorem 34, we se two auxiliary complexes $\boldsymbol{\mathcal{H}}_+ = \{\mathcal{H}_+^{(j)}\}_{j=1}^k$ and $\boldsymbol{\mathcal{H}}_- = \{\mathcal{H}_-^{(j)}\}_{j=1}^k$. Later, in the final phase (see Section 4.4.4), our goal is to apply the Dense Counting Lemma to these auxiliary complexes.

The construction of $\boldsymbol{\mathcal{H}}_+$, $\boldsymbol{\mathcal{H}}_-$ and $\boldsymbol{\mathcal{F}}$ splits into two parts. First (cf. upcoming Lemma 37), we construct an auxiliary (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ which will have the required properties for $1 \le j < k$ (we have $\mathcal{H}_+^{(j)} = \mathcal{H}_-^{(j)} = \mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $1 \le j < k$).

In the second part, we use upcoming Lemma 38 to overcome a 'slight imperfection' of $\mathcal{H}^{(k)}$ and construct $\mathcal{H}_+^{(k)}$, $\mathcal{H}_-^{(k)}$, and $\mathcal{F}^{(k)}$ so that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ (as we will later show in Lemma 40) satisfy the assumptions of the Dense Counting Lemma. Moreover, $\mathcal{H}_+^{(k)}$ and $\mathcal{H}_-^{(k)}$ will "sandwich" $\mathcal{F}^{(k)}$ and $\mathcal{H}^{(k)}$ (i.e. $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$).

We need the following definition in order to state Lemma 37. For a $(j-1)$-uniform hypergraph $\mathcal{H}^{(j-1)}$, we denote by $\hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right) \subseteq \hat{A}(j-1, \boldsymbol{b})$ the set of polyad addresses $\hat{\boldsymbol{x}}^{(j-1)}$ such that

$$\hat{\mathcal{P}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \subseteq \mathcal{H}^{(j-1)} . \tag{33}$$

Lemma 37 (Cleaning Phase II, Part 1)**.** *Given Setup 36, there exists an (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}} = \left\{\mathcal{H}^{(j)}\right\}_{j=1}^k$ such that:*

(*a*) *$\mathcal{H}^{(1)} = \tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$ (and consequently $\hat{A}\left(\mathcal{H}^{(1)}, 1, \boldsymbol{b}\right) = \hat{A}\left(1, \boldsymbol{b}\right)$) and $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$.*

(*b*) *For every $2 \le j < k$, the following holds:*

(*b1*) *For any $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, there is an index set $I\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \subseteq [b_j]$ of size $\left|I\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| = d_j b_j$ such that*

$$\mathcal{H}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\left(\big(\hat{\boldsymbol{x}}^{(j-1)}, \alpha\big)\right) .$$

(*b2*) *For every $j \le i \le \ell$,*

$$\left|\mathcal{K}_i^{(j)}\left(\mathcal{H}^{(j)}\right) \triangle \mathcal{K}_i^{(j)}\left(\tilde{\mathcal{G}}^{(j)}\right)\right| \le \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i .$$

(*c*) *Finally, the (n, ℓ, k)-cylinder $\mathcal{H}^{(k)}$ satisfies the following two properties:*

(*c1*) *For every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\big(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\big)$, $\mathcal{H}^{(k)}$ is $\big(\tilde{\delta}_k, \bar{d}_k(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r}\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ where $\bar{d}_k\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = d_k \pm \sqrt{\delta_k}$.*

(*c2*)

$$\left|\mathcal{K}_\ell^{(k)}\left(\mathcal{H}^{(k)}\right) \triangle \mathcal{K}_\ell^{(k)}\left(\tilde{\mathcal{G}}^{(k)}\right)\right| \le \delta_k^{1/3} \left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell .$$

We prove Lemma 37 in Section 6.2.

Consider the subcomplex $\boldsymbol{\mathcal{H}}^{(\boldsymbol{k-1})} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$. The complex $\boldsymbol{\mathcal{H}}^{(\boldsymbol{k-1})}$ is 'absolutely perfect' by having the following two properties for every $2 \le j < k$:

perfectly equitable (*PE*): For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ and every $\beta \in I(\hat{\boldsymbol{x}}^{(j-1)})$, the $(n/b_1, j, j)$-cylinder $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big)$ is $(\tilde{\delta}_j, \tilde{d}_j, \tilde{r})$-regular w.r.t. its underlying polyad $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

uniformly dense (*UD*): For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$,

$$d\big(\mathcal{H}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = (d_j b_j)(\tilde{d}_j \pm \tilde{\delta}_j) . \tag{34}$$

The property (*PE*) is an immediate consequence of the fact that $\mathscr{P}$ is an almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions. Property (*UD*) easily follows from (*b1*) combined with (*PE*).

We now rewrite the right-hand side of (34) in a more convenient form (cf. (35)). Using (*i*) from Definition 29, we infer

$$d\big(\mathcal{H}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = d_j(1 \pm \tilde{\delta}_j/\tilde{d}_j) \pm b_j \tilde{\delta}_j = d_j \pm (\tilde{\delta}_j/\tilde{d}_j + b_j \tilde{\delta}_j) .$$

As a consequence of Definition 29 (i) and $\tilde{d}_j > \tilde{\delta}_j$, we have $b_j < 2/\tilde{d}_j$. Due to the choice of the constants (cf. Figure 2) $\tilde{\delta}_j \lll \tilde{d}_j$. We therefore infer

$$d\big(\mathcal{H}^{(j)}\big|\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = d_j \pm (\tilde{\delta}_j/\tilde{d}_j + b_j\tilde{\delta}_j) = d_j \pm 3\tilde{\delta}_j/\tilde{d}_j = d_j \pm \sqrt{\tilde{\delta}_j}\,. \tag{35}$$

For each $2 \le j < k$ consider $\mathcal{H}^{(j)}$ as the union

$$\mathcal{H}^{(j)} = \bigcup \big\{\mathcal{H}^{(j)} \cap \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\colon\ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})\big\}\,.$$

From property (PE) and (35) we will infer that $\mathcal{H}^{(j)}$ is $(\tilde{\delta}_j^{1/3}, \tilde{d}_j, 1)$-regular (and, therefore, also $(\varepsilon', \tilde{d}_j, 1)$-regular) w.r.t. $\mathcal{H}^{(j-1)}$ (this will be verified in the proof of Lemma 40 in Section 7.2). This means, however, that the complex $\boldsymbol{\mathcal{H}}^{(k-1)}$ is 'ready' for an application of the Dense Counting Lemma, Theorem 16.

The proof of $(b2)$ is based on the induction assumption (cf. $\mathbf{IHC}_{k-1,\ell}$). The treatment of $\mathcal{H}^{(k)}$ will necessarily have to be different. We shall construct two (n, ℓ, k)-cylinders $\mathcal{H}^{(k)}_+$ and $\mathcal{H}^{(k)}_-$ so that $\mathcal{H}^{(k)}_+ \supseteq \mathcal{H}^{(k)} \supseteq \mathcal{H}^{(k)}_-$. Moreover, we construct $\mathcal{F}^{(k)}$, incomparable with respect to $\mathcal{H}^{(k)}$, but with $\mathcal{H}^{(k)}_+ \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}^{(k)}_-$. To this end, we use the following lemma whose proof we defer to Section 6.3.

Lemma 38 (Cleaning Phase II, Part 2)**.** *Given Setup 36 and the (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}}$ from Part 1 of Cleaning Phase II, Lemma 37, there are (n, ℓ, k)-cylinders $\mathcal{H}^{(k)}_- \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}^{(k)}_+$ such that:*

- (α) $\boldsymbol{\mathcal{H}}_- = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \mathcal{H}^{(k)}_-$, $\boldsymbol{\mathcal{H}}_+ = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \mathcal{H}^{(k)}_+$, *and* $\boldsymbol{\mathcal{F}} = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \mathcal{F}^{(k)}$ *are (n, ℓ, k)-complexes and* $\mathcal{H}^{(k)}_- \subseteq \mathcal{H}^{(k)} \subseteq \mathcal{H}^{(k)}_+$.
- (β) *For every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\left(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\right)$, the following holds:*
  - $(\beta 1)$ $\mathcal{H}^{(k)}_-$ *is* $\big(3\tilde{\delta}_k, d_k - \sqrt{\delta_k}, \tilde{r}\big)$*-regular w.r.t. to* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ *and*
  - $(\beta 2)$ $\mathcal{H}^{(k)}_+$ *is* $\big(3\tilde{\delta}_k, d_k + \sqrt{\delta_k}, \tilde{r}\big)$*-regular w.r.t. to* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$,
  - $(\beta 3)$ $\mathcal{F}^{(k)}$ *is* $\big(21\tilde{\delta}_k, d_k, \tilde{r}\big)$*-regular w.r.t. to* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$.

Cleaning Phase II is now concluded. For future reference, we summarize the effects of Cleaning Phase II.

Setup 39 (After Cleaning Phase II)**.** *Let all constants be chosen as summarized in Figure 2 so that* (29) *and* (31) *hold.*

- *Let $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ be the almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions given after Cleaning Phase I, i.e., after an application of Lemma 30.*
- *Let $\boldsymbol{\mathcal{H}}$ be the (n, ℓ, k)-complex given from Part 1 of Cleaning Phase II, Lemma 37. For every $2 \le j < k$ and $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, let $I(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq [b_j]$ be the index set satisfying (b1) of Lemma 37.*
- *Moreover, let $\boldsymbol{\mathcal{H}}_+$, $\boldsymbol{\mathcal{H}}_-$, and $\boldsymbol{\mathcal{F}}$ be the (n, ℓ, k)-complexes given by Part 2 of Cleaning Phase II, Lemma 38.*

4.4.4. *The Final Phase.* We now finish the proof Theorem 34. The first goal is to show that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ satisfy the assumptions of the Dense Counting Lemma. To this end, we use the upcoming Union Lemma, Lemma 40, stated below. After stating the Union Lemma, we finish the proof of Theorem 34.

Lemma 40 (Union lemma)**.** *Given Setup 39 and $* \in \{+, -\}$, the complex $\boldsymbol{\mathcal{H}}_*$ is $(\boldsymbol{\varepsilon'}, \boldsymbol{d}_*, 1)$-regular where $\boldsymbol{\varepsilon'} = (\varepsilon', \dots, \varepsilon') \in \mathbb{R}^{k-1}$ and $\boldsymbol{d}_* = (d_2^*, \dots, d_k^*)$ with*

$$d_j^* = \begin{cases} d_j & \text{if } \ 2 \le j \le k-1 \\ d_k + \sqrt{\delta_k} & \text{if } \ j = k \text{ and } * = + \\ d_k - \sqrt{\delta_k} & \text{if } \ j = k \text{ and } * = -\,. \end{cases} \tag{36}$$

Similarly, the (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \cup \mathcal{F}^{(k)}$ is $(\boldsymbol{\varepsilon'}, \boldsymbol{d}, 1)$-regular where $\boldsymbol{\varepsilon'} = (\varepsilon', \dots, \varepsilon') \in \mathbb{R}^{k-1}$ and $\boldsymbol{d} = (d_2, \dots, d_k)$.

We give the proof of Lemma 40 in Section 7. We now finish this section with the proof of Theorem 34.

Proof of Theorem 34. Set $\mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $1 \le j < k$ and let $\mathcal{F}^{(k)}$ be given by Lemma 38. Consequently, $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$ is an (n, ℓ, k)-complex and Lemma 40 gives (i) of Theorem 34. Moreover, due to part (a) of

Lemma 37, we have $\mathcal{G}^{(1)} = \mathcal{H}^{(1)} = \mathcal{F}^{(1)}$ and $\mathcal{G}^{(2)} = \mathcal{H}^{(2)} = \mathcal{F}^{(2)}$ which yields (*ii*) of Theorem 34. It is left to verify part (*iii*) of the theorem.

As an intermediate step, we first consider $\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})$. Since $\mathcal{H}_+^{(k)} \supseteq \mathcal{H}^{(k)} \cup \mathcal{F}^{(k)}$ and $\mathcal{H}^{(k)} \cap \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$ (cf. Lemma 38), we have

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_+^{(k)}) \setminus \mathcal{K}_\ell^{(k)}(\mathcal{H}_-^{(k)})\right|. \tag{37}$$

We infer from Lemma 40 and the choice of ε' in (21) and n_0 in (28) that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ satisfy the assumptions of the Dense Counting Lemma, Theorem 16. Consequently,

$$\begin{aligned}\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_+^{(k)})\right| &\leq \left(1+\sqrt{\delta_k}\right)\left(d_k+\sqrt{\delta_k}\right)^{\binom{\ell}{k}} \prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}} \times n^\ell \leq \left(1+\sqrt{\delta_k}\right)\left(1+2\binom{\ell}{k}\frac{\sqrt{\delta_k}}{d_k}\right)\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell \\ &\leq \left(1+\delta_k^{1/3}\right)\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell.\end{aligned} \tag{38}$$

Similarly,

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_-^{(k)})\right| \geq \left(1-\delta_k^{1/3}\right)\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{39}$$

Therefore, from (37), (38) and (39), we infer

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq 2\delta_k^{1/3}\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{40}$$

We now prove (*iii*) of Theorem 34. Using the triangle-inequality, we infer

$$\begin{aligned}\left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq &\left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right| + \\ &+\left|\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)})\right| + \\ &+\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right|.\end{aligned} \tag{41}$$

Then (32), (*c2*) of Lemma 37, and (40) bound the right-hand side of (41) and, hence,

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq \left(\delta_k + 3\delta_k^{1/3}\right)\prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{42}$$

Part (*iii*) of Theorem 34 now follows from $\gamma \ggg \delta_k$ (cf. Figure 2). This concludes the proof of Theorem 34. □

5. Proof of Cleaning Phase I

The proof of Lemma 30 is organized as follows. We first fix all constants involved in the proof (as usual). We then inductively construct the almost perfect family of partitions $\mathscr{P}$ and the complex $\tilde{\boldsymbol{\mathcal{G}}}$ promised by Lemma 30. Finally, we verify that $\mathscr{P}$ and $\tilde{\boldsymbol{\mathcal{G}}}$ have the desired properties.

5.1. **Constants.** Let $\boldsymbol{d} = (d_2, \dots, d_k)$ be a vector of positive reals and let $\delta_3, \dots, \delta_k$ satisfy $0 \leq \delta_j \leq d_j/2$ for $j = 3, \dots, k$. Moreover, let $\tilde{\delta}_k$ be a positive real and let $\tilde{\boldsymbol{\delta}}(\boldsymbol{D})$ and $\tilde{r}(B_1, \boldsymbol{D})$ be the arbitrary positive functions in variables $\boldsymbol{D} = (D_2, \dots, D_{k-1})$ and B_1 given by the lemma. The proof of Lemma 30 relies on the Regularity Lemma, and more specifically, on Corollary 28. The proof also relies on the Induction Hypothesis on the Counting Lemma ($\mathbf{IHC}_{k-1,\ell}$), Statement 32, with $\ell = k$. Therefore, for the proof of Lemma 30 presented here, we assume that $\mathbf{IHC}_{k-1,\ell}$ holds (cf. (9)).

We set

$$\eta = \frac{1}{4}, \quad \sigma_j = \begin{cases} d_2 & \text{if} \quad j = 2 \\ 1 & \text{if} \quad 3 \leq j \leq k-1 \end{cases} \quad \text{and} \quad \delta_k' = \mu = \frac{\tilde{\delta}_k}{2^{\ell+k}}\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\,. \tag{43}$$

We also fix functions in variables $D_j, \ldots, D_{k-1}$ for $j = k-1, \ldots, 2$ so that

$$\begin{aligned}\delta'_j\left(D_j, \ldots, D_{k-1}\right) &< \min\left\{\frac{D_j}{18}\tilde{\delta}_j\left(D_j, \ldots, D_{k-1}\right), \frac{D_j^3}{36}\right\} < \frac{D_j^2}{9}\\ \delta'_j\left(D_j, \ldots, D_{k-1}\right) &< \frac{D_j}{9}\delta_j\big(\mathbf{IHC}_{k-1,\ell}(\eta, D_{k-1}, \delta'_{k-1}(D_{k-1}), D_{k-2}, \ldots, \delta'_{j+1}(D_{j+1}\ldots, D_{k-1}), D_j)\big)\,.\end{aligned} \tag{44}$$

(Observe that the right-hand side of the last inequality is a function in variables $D_j, \ldots, D_{k-1}$.) Similarly, we set

$$\begin{aligned}&r'\left(B_1, \boldsymbol{D}\right) \geq \tilde{r}\left(B_1, \boldsymbol{D}\right),\\ &r'\left(B_1, \boldsymbol{D}\right) \geq r\Big(\mathbf{IHC}_{k-1,\ell}\left(\eta, D_{k-1}, \delta'_{k-1}(D_{k-1}), D_{k-2}, \ldots, \delta'_{j+1}(D_{j+1}\ldots, D_{k-1}), D_j\right)\Big)\end{aligned} \tag{45}$$

where, without loss of generality, we may assume that the functions given in (44) and (45) are monotone. Corollary 28 then yields the integer constants n_k and L_k. Next we define the constants promised by Lemma 30 as follows

$$\begin{aligned}&\tilde{c}_j = \frac{1}{2^{\ell+2}L_k^k} \text{ for } j = 2, \ldots, k-1\,, \quad \tilde{\boldsymbol{c}} = (\tilde{c}_2, \ldots, \tilde{c}_{k-1})\,,\\ &\tilde{L}_k = 2^{\ell+k^2}L_k^{k-1}\prod_{j=2}^{k-1}\left(\frac{1}{\tilde{c}_j}\right)^{\binom{k-1}{j}} \text{ and } \quad \delta_2 = \frac{\delta'_2(\tilde{\boldsymbol{c}})}{\tilde{L}_k^2}\,.\end{aligned} \tag{46}$$

Finally, let $m_{k-1,\ell}$ be the integer given by Statement 32 applied to the constants η, $\tilde{c}_{k-1}$, $\delta'_{k-1}(\tilde{c}_{k-1}), \ldots, \tilde{c}_2$, $\delta'_2(\tilde{\boldsymbol{c}})$ and set $\tilde{n}_k = \max\{n_k, L_k m_{k-1,\ell}\}$.

5.2. **Getting started.** Let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ be an (n, ℓ, k)-complex with $n \geq \tilde{n}_k$. We apply Corollary 28 to $\boldsymbol{\mathcal{G}}$ to obtain a $\big(\mu, \boldsymbol{\delta'}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, r'(\tilde{\boldsymbol{d}})\big)$-equitable $\big(\delta'_k, r'(\tilde{\boldsymbol{d}})\big)$-regular family of partitions $\mathscr{R} = \mathscr{R}\left(k-1, \boldsymbol{a}, \boldsymbol{\varphi}\right) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ refining $\boldsymbol{\mathcal{G}}$ (cf. Definition 27) where $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \ldots, \tilde{d}_{k-1})$ is the density vector of the partition $\mathscr{R}$. Note that it follows from our choice of σ_j in (43) that

$$d_2/\tilde{d}_2 \textit{ and, for all } j = 3, \ldots, k-1,\ 1/\tilde{d}_j, \textit{ are integers.} \tag{47}$$

We now make a few preparations concerning notation. Having $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \ldots, \tilde{d}_{k-1})$ as an outcome of Corollary 28, we derive the constants δ'_j, $\tilde{\delta}_j$ for $j = 2, \ldots, k-1$ and r' and $\tilde{r}$ from the functions given in (44) and (45) by setting

$$\delta'_j = \delta'_j(\tilde{d}_j, \ldots, \tilde{d}_{k-1}) < \tilde{\delta}_j = \tilde{\delta}_j(\tilde{d}_j, \ldots, \tilde{d}_{k-1}) \quad \text{and} \quad r' = r'(a_1, \tilde{\boldsymbol{d}}) \geq \tilde{r}(a_1, \tilde{\boldsymbol{d}}) = \tilde{r}\,,$$

(the inequalities above follow immediately from (44) and (45)). Moreover, we set $\boldsymbol{\delta'} = (\delta'_2, \ldots, \delta'_{k-1})$ and $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \ldots, \tilde{\delta}_{k-1})$.

For any $j = 2, \ldots, k-1$ and $\hat{\boldsymbol{y}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$, let $a_j^{\text{reg}} = a_j^{\text{reg}}(\hat{\boldsymbol{y}}^{(j-1)})$ be the number of $(\delta'_j, \tilde{d}_j, r')$-regular $(n/a_1, j, j)$-cylinders belonging to $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{y}}^{(j-1)})$. We then observe that

$$a_j^{\text{reg}} = a_j^{\text{reg}}(\hat{\boldsymbol{y}}^{(j-1)}) \leq \frac{1}{\tilde{d}_j - \delta'_j} \leq \frac{2}{\tilde{d}_j}\,. \tag{48}$$

Finally, we fix the integer vector $\boldsymbol{b} = (b_1, \ldots, b_{k-1})$. We set

$$b_1 = a_1, \quad b_2 = \left\lceil \frac{1}{\tilde{d}_2 + 9\delta'_2/\tilde{d}_2} \right\rceil \leq \frac{2}{\tilde{d}_j}\,, \text{ and } \quad b_j \overset{(47)}{=} \frac{1}{\tilde{d}_j} \text{ for } j = 3, \ldots, k-1\,. \tag{49}$$

We then define $\bar{\boldsymbol{b}} = (b_1, b_2+1, b_3, \ldots, b_{k-1})$.

Before constructing the promised almost perfect $\big(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b}\big)$-family of partitions $\mathscr{P} = \mathscr{P}\left(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi}\right) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ (cf. Definition 29) and the (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$, we proceed with the following simple observation.

Observation regarding 'bad' j-tuples. Since $\mathscr{R}$ is a $(\mu, \boldsymbol{\delta'}, \tilde{\boldsymbol{d}}, r')$-equitable (δ'_k, r')-regular partition, all but at most μn^k crossing (w.r.t. $\mathcal{G}^{(1)}$) k-tuples belong to $(\boldsymbol{\delta'}, \tilde{\boldsymbol{d}}, r')$-regular $(n/a_1, k, k-1)$-complexes $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(k-1)}) = \{\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(k-1)})\}_{j=1}^{k-1}$ given by the family of partitions $\mathscr{R}$. We assert that

$$\begin{gathered}\text{for each } 2 \le j \le k, \text{ at most } \mu\binom{k}{j} n^j \text{ crossing } j\text{-tuples belong to} \\ \big((\delta'_2, \dots, \delta'_{j-1}), (\tilde{d}_2, \dots, \tilde{d}_{j-1}), r'\big)\text{-irregular } (n/a_1, j, j-1)\text{-complexes of } \mathscr{R}.\end{gathered} \tag{50}$$

Indeed, a j-tuple belonging to an irregular $(n/a_1, j, j-1)$-complex can be extended to $\binom{l-j}{k-j} n^{k-j}$ crossing k-tuples and at most $\binom{k}{j}$ such j-tuples can be extended to the same k-tuple. Each such k-tuple necessarily belongs to an irregular $(n/a_1, k, k-1)$-complex.

Itinerary. We define complex $\tilde{\boldsymbol{\mathcal{G}}}$ and family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ so that $\mathscr{P}$ is an almost perfect family of partitions refining $\tilde{\boldsymbol{\mathcal{G}}}$. Our plan is to alter the family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ into the family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi}) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$. The families $\mathscr{P}$ and $\mathscr{R}$ will overlap in the regular elements of $\mathscr{R}$. The elements of $\mathscr{R}$ which are not regular are substituted by random cylinders.

We construct $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ inductively for $j = 1, \dots, k-1$. First set $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$. Since $b_1 = a_1$, we have $A(1, \boldsymbol{a}) = A(1, \boldsymbol{b}) = A(1, \bar{\boldsymbol{b}})$ and $\hat{A}(1, \boldsymbol{a}) = \hat{A}(1, \boldsymbol{b}) = \hat{A}(1, \bar{\boldsymbol{b}})$. We set $\psi_1 \equiv \varphi_1$ and define $\mathscr{P}^{(1)} = \mathscr{R}^{(1)}$. In other words, both $\mathscr{R}$ and $\mathscr{P}$ split the sets V_λ for $\lambda \in [\ell]$ into the same pieces $V_\lambda = V_{\lambda,1} \cup \dots \cup V_{\lambda, b_1}$.

For $2 \le j < k$, we shall define $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ in such a way that the following statement $(\mathcal{C}_j)$ holds:

$(\mathcal{C}_j)$ There is a partition $\mathscr{P}^{(j)} = \mathscr{P}^{(j)}_{\text{orig}} \cup \mathscr{P}^{(j)}_{\text{new}}$ of $K^{(j)}_\ell(V_1, \dots, V_\ell)$ where, for $* \in \{\text{orig}, \text{new}\}$, we define

$$\mathcal{P}^{(j)}_* = \bigcup \{\mathcal{P}^{(j)} \colon \mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_*\},$$

and an (n, ℓ, j)-cylinder $\tilde{\mathcal{G}}^{(j)} \subseteq K^{(j)}_\ell(V_1, \dots, V_\ell)$ such that (I)–(III) below hold:

$$\begin{aligned}\text{(I) } \mathscr{P}^{(j)}_{\text{orig}} = \big\{&\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}) \colon \boldsymbol{y}^{(j)} \in A(j, \boldsymbol{a}) \text{ and } \boldsymbol{\mathcal{R}}^{(j)}(\boldsymbol{y}^{(j)}) = \{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)})\}_{h=1}^{j} \\ &\text{is a } \big((\delta'_2, \dots, \delta'_j), (\tilde{d}_2, \dots, \tilde{d}_j), r'\big)\text{-regular } (n/a_1, j, j)\text{-complex}\big\},\end{aligned}$$

$$\text{(II) } \tilde{\mathcal{G}}^{(j)} = \begin{cases} \mathcal{G}^{(2)} & \text{if} \quad j = 2 \\ \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\text{orig}} = \mathcal{G}^{(j)} \setminus \mathcal{P}^{(j)}_{\text{new}} & \text{if} \quad 3 \le j < k \end{cases} \qquad \text{and}$$

(III) the family of partitions $\mathscr{P}_j = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(j)}\}$ is an almost perfect $\big((9\delta'_2/\tilde{d}_2, \dots, 9\delta'_j/\tilde{d}_j), (\tilde{d}_2, \dots, \tilde{d}_j), r', \boldsymbol{b}\big)$-family.

Before we give an inductive proof of statement $(\mathcal{C}_j)$, we list a few of its consequences in Fact 41. The properties *(1)*–*(5)* of Fact 41 will be derived directly from $(\mathcal{C}_j)$. They are utilized in our proof, in particular, we use Fact 41 with $j-1$ to establish $(\mathcal{C}_j)$.

Fact 41 (Consequences of $(\mathcal{C}_j)$)**.** *Let $2 \le j \le k-1$ be fixed. If $(\mathcal{C}_{j'})$ holds for $2 \le j' \le j$ and if $\mathscr{P}^{(2)}$ refines $\tilde{\mathcal{G}}^{(2)}$, then the following is true:*

(1) $\tilde{\mathcal{G}}^{(j)} \subseteq \mathcal{G}^{(j)}$,

(2) $\tilde{\boldsymbol{\mathcal{G}}}^{(\boldsymbol{j})} = \{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^{j}$ *is an (n, ℓ, j)-complex and for each $2 \le h \le j$,* $\tilde{\mathcal{G}}^{(h)} \subseteq \mathcal{K}^{(h-1)}_h\big(\tilde{\mathcal{G}}^{(h-1)}\big)$,

(3) $\mathscr{P}_j$ *refines the complex* $\tilde{\boldsymbol{\mathcal{G}}}^{(\boldsymbol{j})}$,

(4) for every $j \le i \le \ell$,

$$\Big|\mathcal{K}^{(j)}_i(\tilde{\mathcal{G}}^{(j)}) \triangle \mathcal{K}^{(j)}_i(\mathcal{G}^{(j)})\Big| \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,, \qquad \textit{and}$$

(5) for every $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \boldsymbol{b})$,

$$\Big|\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)\Big| = (1 \pm \eta) \prod_{h=2}^{j} \tilde{d}_h^{\binom{j+1}{h}} \times \left(\frac{n}{b_1}\right)^{j+1} > \frac{(n/b_1)^{j+1}}{\ln(n/b_1)}.$$

Proof of Fact 41. Part (*1*) follows clearly from (II). We prove (*2*) by induction on j. For $j = 1$ or 2 there is nothing to prove. Let $j \geq 3$. Suppose $(\mathcal{C}_i)$ is true for $2 \leq i \leq j$ and suppose, by induction, (*2*) holds for $j-1$, i.e, $\tilde{\boldsymbol{\mathcal{G}}}^{(\boldsymbol{j-1})}$ is an $(n, \ell, j-1)$-complex. We show that every j-tuple $J \in \tilde{\mathcal{G}}^{(j)}$ satisfies $J \in \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$.

Let $J \in \tilde{\mathcal{G}}^{(j)}$ be fixed. It then follows from (II) of $(\mathcal{C}_j)$ that

$$J \in \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\text{orig}} . \tag{51}$$

We first confirm

$$J \in \mathcal{K}_j^{(j-1)}(\mathcal{P}^{(j-1)}_{\text{orig}}) . \tag{52}$$

To that end, since $J \in \mathcal{P}^{(j)}_{\text{orig}}$, it follows from (I) of $(\mathcal{C}_j)$ that there exists $\boldsymbol{y}^{(j)} \in A(j, \boldsymbol{a})$ such that $J \in \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)})$ and the complex $\boldsymbol{\mathcal{R}}^{(j)}(\boldsymbol{y}^{(j)}) = \{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)})\}_{h=1}^{j}$ is $\big((\delta_2', \dots, \delta_j'), (\tilde{d}_2, \dots, \tilde{d}_j), r'\big)$-regular. Consequently, $J \in \mathcal{K}_j^{(j-1)}\big(\mathcal{R}^{(j-1)}(\boldsymbol{y}^{(j)})\big)$ and by (I) of $(\mathcal{C}_{j-1})$ we have that $\mathcal{R}^{(j-1)}(\boldsymbol{y}^{(j)}) \subseteq \mathcal{P}^{(j-1)}_{\text{orig}}$. This yields $J \in \mathcal{K}_j^{(j-1)}(\mathcal{P}^{(j-1)}_{\text{orig}})$ as claimed in (52).

Now from (51) and (52), we infer that $J \in \mathcal{K}_j^{(j-1)}(\mathcal{G}^{(j-1)} \cap \mathcal{P}^{(j-1)}_{\text{orig}})$ (since $\boldsymbol{\mathcal{G}}$ is a complex), and so by (II) of $(\mathcal{C}_{j-1})$ we have $J \in \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. This completes the proof of (*2*).

Next we show part (*3*), again by induction on j. The statement is trivial for $j = 1$. It holds for $j = 2$ by assumption of Fact 41. So let $j \geq 3$ and assume that $\mathscr{P}_{j-1}$ refines $\{\tilde{\mathcal{G}}^{(h)}\}_{j=1}^{j-1}$. We have to show that either $\mathcal{P}^{(j)} \subseteq \tilde{\mathcal{G}}^{(j)}$ or $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$ for every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$. So let $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ be fixed. If $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\text{new}}$, then $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$ by (II) of $(\mathcal{C}_j)$. Therefore, we may assume that $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\text{orig}}$. Now, if $\mathcal{P}^{(j)} \cap \mathcal{G}^{(j)} = \varnothing$, then again by (II) of $(\mathcal{C}_j)$ we infer $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$. On the other hand, if $\mathcal{P}^{(j)} \cap \mathcal{G}^{(j)} \neq \varnothing$, then $\mathcal{P}^{(j)} \subseteq \mathcal{G}^{(j)}$ since $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\text{orig}}$, (I) of $(\mathcal{C}_j)$, and the fact that the original family of partitions $\mathscr{R}$ refines the complex $\boldsymbol{\mathcal{G}}$. Therefore, $\mathcal{P}^{(j)} \subseteq \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\text{orig}} = \tilde{\mathcal{G}}^{(j)}$ by (*1*) of Fact 41. This verifies (*3*) of Fact 41.

Next we show (*4*) of Fact 41. From (50) and (I) and (II) of $(\mathcal{C}_j)$ we infer that

$$\big|\mathcal{G}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}\big| = \big|\mathcal{G}^{(j)} \setminus \tilde{\mathcal{G}}^{(j)}\big| \leq \mu \binom{k}{j} n^j .$$

Consequently, by the choice of μ in (43)

$$\Big|\mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\mathcal{G}^{(j)})\Big| \leq \mu \binom{k}{j} n^j \times \binom{\ell - j}{i - j} n^{i-j} \leq \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i ,$$

which yields (*4*).

Finally, we note that (*5*) follows from (III) and $\mathbf{IHC}_{k-1,\ell}$ (cf. (9)) since $j \leq k-1$. In particular, (*5*) is a consequence of the choice of δ_j' and r' in (44) and (45), (III) of $(\mathcal{C}_{k-1})$, and (9). ☐

5.3. **Proof of Statement** $(\mathcal{C}_j)$**.** As mentioned earlier, we verify $(\mathcal{C}_j)$ by induction on j.

5.3.1. *The Induction Start.* In the immediate sequel, we define $\mathscr{P}^{(2)} = \mathscr{P}^{(2)}_{\text{new}} \cup \mathscr{P}^{(2)}_{\text{orig}}$ of $K_\ell^{(2)}(V_1, \dots, V_\ell)$. In our construction, we use that due to (44) and (46), our constants satisfy

$$a_1^2 \delta_2 < L_k^2 \delta_2 < \tilde{L}_k^2 \delta_2 < \delta_2' < \tilde{d}_2 \leq d_2 \tag{53}$$

and also use that $d_2/\tilde{d}_2$ is an integer (see (47)). Before constructing the partition $\mathscr{P}^{(2)}$, we require some notation.

Notation. Recall that the partition $\mathscr{R}^{(2)} = \{\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) \colon \boldsymbol{y}^{(2)} \in A(2, \boldsymbol{a})\}$ refines the partition $\mathcal{G}^{(2)} \cup \overline{\mathcal{G}^{(2)}}$ (here, $\overline{\mathcal{G}^{(2)}} = K_\ell^{(2)}(V_1, \dots, V_\ell) \setminus \mathcal{G}^{(2)}$). Therefore, for each $\hat{\boldsymbol{y}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a})$, there exist disjoint sets of indices $I_2^{\text{reg}} = I_2^{\text{reg}}(\hat{\boldsymbol{y}}^{(1)})$ and $\overline{I}_2^{\text{reg}} = \overline{I}_2^{\text{reg}}(\hat{\boldsymbol{y}}^{(1)})$ so that $\big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\big\}_{\alpha \in I_2^{\text{reg}}}$ and $\big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\big\}_{\alpha \in \overline{I}_2^{\text{reg}}}$ are the collections of all $(\delta_2', \tilde{d}_2, 1)$-regular graphs $\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) = \mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)$ whose edge sets are subsets of $\mathcal{G}^{(2)}(\boldsymbol{y}^{(1)}) = \mathcal{G}^{(2)}[V_{\lambda,\beta} \cup V_{\lambda',\beta'}]$ and $\overline{\mathcal{G}^{(2)}}(\boldsymbol{y}^{(1)}) = V_{\lambda,\beta} \times V_{\lambda',\beta'} \setminus \mathcal{G}^{(2)}$, respectively.

Plan for constructing $\mathscr{P}^{(2)}$**.** We now outline our plan for constructing $\mathscr{P}^{(2)} = \{\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)}) \colon \boldsymbol{x}^{(2)} \in A(2, \bar{\boldsymbol{b}})\}$. Later we fill in the technical details. With $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a}) = \hat{A}(1, \bar{\boldsymbol{b}})$

fixed, we define a partition $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ of $\mathcal{K}_2^{(1)}\big(\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)})\big) = V_{\lambda,\beta} \times V_{\lambda',\beta'}$. More precisely, with $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)}$ defining a pair of sets $V_{\lambda,\beta}$, $V_{\lambda',\beta'}$, we consider all regular subgraphs of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$ from the partition $\mathscr{R}^{(2)}$ and leave them in the "original part" ($\mathscr{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)})$) of $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$. In other words, for $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)}$ we set

$$\mathscr{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)}) = \Big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)},\alpha)\big)\Big\}_{\alpha\in I_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)})} \cup \Big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)},\alpha)\big)\Big\}_{\alpha\in \overline{I}_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)})}. \tag{54}$$

This collection of graphs consist of all subgraphs of $V_{\lambda,\beta}\times V_{\lambda',\beta'}$ belonging to $\mathscr{R}^{(2)}$ which are $(\delta_2', \tilde{d}_2, 1)$-regular. In order to simplify the notation, we set

$$\mathcal{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)}) = \bigcup \Big\{\mathcal{P}^{(2)}\colon\ \mathcal{P}^{(2)} \in \mathscr{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)})\Big\}\,.$$

For the construction of the partition of $V_{\lambda,\beta} \times V_{\lambda',\beta'} \setminus \mathcal{P}^{(2)}_{\rm orig}$, we will use the Slicing Lemma to introduce new $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular graphs that do not belong to $\mathscr{R}^{(2)}$. We shall call the collection of those graphs $\mathscr{P}^{(2)}_{\rm new}(\hat{\boldsymbol{x}}^{(1)})$. We now provide the technical details to the plan described above.

Technical details for constructing $\mathscr{P}^{(2)}$. Let $\hat{\boldsymbol{x}}^{(1)} = \big((\lambda,\lambda'),(\beta,\beta')\big) \in \hat{A}(1,\boldsymbol{a})$ remain fixed. Let

$$\mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)}) \cap \mathcal{G}^{(2)}$$

be the union of all graphs $\mathcal{P}^{(2)} \subseteq \mathcal{G}^{(2)}$ in $\mathscr{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)})$. Similarly, we define

$$\overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)}) \cap \overline{\mathcal{G}^{(2)}}\,.$$

Note that while $\overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ and $\mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$ are disjoint, they are not not necessarily complements of each other. Moreover, observe that $\mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$ is the union of $\alpha_2^{\rm reg} = |I_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)})| \le a_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)}) \le 2/\tilde{d}_2$ (see (48)) $(\delta_2', \tilde{d}_2, 1)$-regular graphs. Consequently, $\mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \alpha_2^{\rm reg}\tilde{d}_2, 1)$-regular (cf. Proposition 47). Similarly, $\overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \overline{\alpha}_2^{\rm reg}\tilde{d}_2, 1)$-regular, where $\overline{\alpha}_2^{\rm reg} = |\overline{I}_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)})|$.

Since $\mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular by the assumption of Lemma 30, we infer that $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{G}^{(2)}[V_{\lambda,\beta}\cup V_{\lambda',\beta'}]$ is $(a_1^2\delta_2, d_2, 1)$-regular. Therefore, $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ is $(\delta_2', d_2, 1)$-regular by (53). Consequently, since $2\delta_2'/\tilde{d}_2 + \delta_2' \le 3\delta_2'/\tilde{d}_2$ we have that $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$ is $(3\delta_2'/\tilde{d}_2, d_2 - \overline{\alpha}_2^{\rm reg}\tilde{d}_2, 1)$-regular. We now apply the Slicing Lemma, Lemma 31, to $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$.

To this end, recall $d_2/\tilde{d}_2$ is an integer (see (47)) and set $p = \tilde{d}_2(d_2 - \alpha_2^{\rm reg}\tilde{d}_2)^{-1}$ so that $1/p = d_2/\tilde{d}_2 - \alpha_2^{\rm reg}$ is an integer. We apply the Slicing Lemma with $\varrho = d_2 - \alpha_2^{\rm reg}\tilde{d}_2$, $\delta = 3\delta_2'/\tilde{d}_2$, p as above and $r_{\rm SL} = 1$ to decompose $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})\setminus\mathcal{G}^{(2)}_{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$ into $1/p = d_2/\tilde{d}_2 - \alpha_2^{\rm reg}$ pairwise edge-disjoint $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular graphs. Denote the family of these bipartite graphs by $\mathscr{P}^{(2)}_{{\rm new},\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$.

The partition $\mathscr{P}^{(2)}_{{\rm new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$ of $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})\setminus\overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ will be defined similarly. Indeed, the graph $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$ is $(a_1^2\delta_2, 1-d_2, 1)$-regular since it is the complement of the $(a_1\delta_2, d_2, 1)$-regular graph $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$. By (53), the graph $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$ is then also $(\delta_2'/\tilde{d}_2, 1-d_2, 1)$-regular. Furthermore, $\overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \overline{\alpha}_2^{\rm reg}\tilde{d}_2, 1)$-regular (since $\overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ is the union of $\overline{\alpha}_2^{\rm reg}$ disjoint $(\delta_2', \tilde{d}_2, 1)$-regular graphs and $\overline{\alpha}_2^{\rm reg} \le a_2^{\rm reg} \le 2/\tilde{d}_2$ by (48)). Consequently, $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(3\delta_2'/\tilde{d}_2, 1 - d_2 - \overline{\alpha}_2^{\rm reg}\tilde{d}_2, 1)$-regular.

We apply the Slicing Lemma with $\varrho = 1 - d_2 - \overline{\alpha}_2^{\rm reg}\tilde{d}_2$, $\delta = 3\delta_2'/\tilde{d}_2$, $p = \tilde{d}_2/\varrho$ and $r_{\rm SL} = 1$ to decompose $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\rm reg}}(\hat{\boldsymbol{x}}^{(1)})$ into a family $\mathscr{P}^{(2)}_{{\rm new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$ of bipartite graphs. We conclude that all but at most one of which are $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular. Indeed, note that since (47) guaranteed that $d_2/\tilde{d}_2$ is an integer, we are unable to ensure that $1/p = (1 - d_2 - \overline{\alpha}_2^{\rm reg}\tilde{d}_2)/\tilde{d}_2$ is an integer as well. Consequently, the application of the Slicing Lemma may admit at most one sparse bipartite graph.

For $\hat{\boldsymbol{x}}^{(1)} = \big((\lambda,\lambda'),(\beta,\beta')\big)$, set

$$\mathscr{P}^{(2)}_{\rm new}(\hat{\boldsymbol{x}}^{(1)}) = \mathscr{P}^{(2)}_{{\rm new},\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \cup \mathscr{P}^{(2)}_{{\rm new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$$

and

$$\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \mathscr{P}^{(2)}_{\rm new}(\hat{\boldsymbol{x}}^{(1)}) \cup \mathscr{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)})\,.$$

Also set $z(\hat{\boldsymbol{x}}^{(1)}) = |\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})|$. The partition $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ has the following properties:

(A) $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ is a partition of $V_{\lambda,\beta}\times V_{\lambda',\beta'}$.

(B) $z(\hat{\boldsymbol{x}}^{(1)}) \in \{b_2, b_2+1\}$. Indeed, since all graphs but at most 1 from $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ have density within $\tilde{d}_2 \pm 9\delta_2'/\tilde{d}_2$, it therefore follows that

$$\frac{1}{\tilde{d}_2 + 9\delta_2'/\tilde{d}_2} \le z(\hat{\boldsymbol{x}}^{(1)}) \le \frac{1}{\tilde{d}_2 - 9\delta_2'/\tilde{d}_2} + 1. \tag{55}$$

It follows from (44) that $9\delta_2'/\tilde{d}_2 < (\tilde{d}_2/2)^2$ yielding $(\tilde{d}_2 - 9\delta_2'/\tilde{d}_2)^{-1} - (\tilde{d}_2 + 9\delta_2'/\tilde{d}_2)^{-1} < 1$. Consequently, $z(\hat{\boldsymbol{x}}^{(1)}) \in \{b_2, b_2+1\}$ follows from (49).

(C) $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ refines $\mathcal{G}^{(2)} = \tilde{\mathcal{G}}^{(2)}$ in the sense that for every $\alpha \in [z(\hat{\boldsymbol{x}}^{(1)})]$ either $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)},\alpha)\big) \subseteq \mathcal{G}^{(2)}$ or $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)},\alpha)\big) \cap \mathcal{G} = \varnothing$.

(D) All graphs but at most one from $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ are $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular. Moreover, the exceptional graph belongs to $\mathscr{P}^{(2)}_{{\rm new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)}) \subseteq \mathscr{P}^{(2)}_{\rm new}(\hat{\boldsymbol{x}}^{(1)})$ and we may assume with an appropriate addressing the exceptional graph is always $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)}, b_2+1)\big)$.

Now, we set

$$\tilde{\mathcal{G}}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \bigcup\Big\{\mathcal{P}^{(2)} \in \mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})\colon\ \mathcal{P}^{(2)} \subseteq \mathcal{G}^{(2)}\Big\},\ \ \tilde{\mathcal{G}}^{(2)} = \bigcup\Big\{\tilde{\mathcal{G}}^{(2)}(\hat{\boldsymbol{x}}^{(1)})\colon\ \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\bar{\boldsymbol{b}}) = \hat{A}(1,\boldsymbol{a})\Big\} \tag{56}$$

and we set

$$\mathscr{P}^{(2)}_{\rm new} = \bigcup\Big\{\mathscr{P}^{(2)}_{\rm new}(\hat{\boldsymbol{x}}^{(1)})\colon\ \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\bar{\boldsymbol{b}})\Big\}\,,$$

$$\mathscr{P}^{(2)}_{\rm orig} = \bigcup\Big\{\mathscr{P}^{(2)}_{\rm orig}(\hat{\boldsymbol{x}}^{(1)})\colon\ \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\bar{\boldsymbol{b}})\Big\} \quad\text{and}\quad \mathscr{P}^{(2)} = \mathscr{P}^{(2)}_{\rm new} \cup \mathscr{P}^{(2)}_{\rm orig}\,.$$

It is left to verify (I)–(III) of the statement $(\mathcal{C}_2)$. Due to (54) and the definition of $I_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$ and $\overline{I}_2^{\rm reg}(\hat{\boldsymbol{x}}^{(1)})$, for every $\hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\bar{\boldsymbol{b}})$, we infer that

$$\mathscr{P}^{(2)}_{\rm orig} = \Big\{\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)})\colon\ \mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) \text{ is } (\delta_2', \tilde{d}_2, 1)\text{-regular}\Big\}\,,$$

which yields (I) of $(\mathcal{C}_2)$. Owing to (C) from above and (56), we have $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ (which is (II)) and

$$\mathscr{P}^{(2)} \text{ refines } \tilde{\mathcal{G}}^{(2)}\,. \tag{57}$$

Finally, from (B) (cf. (55)) and $\delta_2' \le \tilde{d}_2\tilde{\delta}_2/18$ (cf. (44)), we infer

$$1 - \frac{\tilde{\delta}_2}{\tilde{d}_2} \overset{(44)}{\le} \frac{\tilde{d}_2^2}{\tilde{d}_2^2 + 9\delta_2'} \overset{(55)}{\le} \tilde{d}_2 b_2 \overset{(55)}{\le} \frac{\tilde{d}_2^2}{\tilde{d}_2^2 - 9\delta_2'} \overset{(44)}{\le} 1 + \frac{\tilde{\delta}_2}{\tilde{d}_2}\,. \tag{58}$$

Now, (58) and (D) yield that $\mathscr{P}_2 = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ is an almost perfect $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, r', \boldsymbol{b})$-family (see Definition 29), which gives (III) of $(\mathcal{C}_2)$.

We again remind the reader that we choose the addressing of the partition classes $\mathscr{P}^{(2)}$ in such a way that for each $\boldsymbol{x}^{(2)} \in A(2,\boldsymbol{b})$, the graph $\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)})$ is $(9\delta_2',/\tilde{d}_2, \tilde{d}_2, r')$-regular. The graph $\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)})$ may not be $(9\delta_2',/\tilde{d}_2, \tilde{d}_2, r')$-regular if and only if $\boldsymbol{x}^{(2)} \in A(2,\bar{\boldsymbol{b}}) \setminus A(2,\boldsymbol{b})$.

This concludes the construction of $\mathscr{P}^{(2)}$ which satisfies $(\mathcal{C}_2)$ and, therefore, we established the induction start of our construction of $\mathscr{P}$ and $\tilde{\boldsymbol{\mathcal{G}}}$. Also note that we additionally verified (57).

5.3.2. *The Inductive Step.* We proceed to the inductive step and construct partition $\mathscr{P}^{(j+1)}$ and $(n,\ell,j+1)$-cylinder $\tilde{\mathcal{G}}^{(j+1)}$ which will satisfy (I)–(III) of $(\mathcal{C}_{j+1})$. Moreover, we assume that $\mathscr{P}^{(h)}$ and $\tilde{\mathcal{G}}^{(h)}$ satisfying $(\mathcal{C}_h)$, $2 \le h \le j$, are given. Moreover, due to (57), we assume Fact 41 holds as well for $2 \le h \le j$.

Our work in constructing $\mathscr{P}^{(j+1)}$ will be quite similar, albeit easier, than our work for constructing $\mathscr{P}^{(2)}$. This is in part because we do not require that $\tilde{\mathcal{G}}^{(j+1)} = \mathcal{G}^{(j+1)}$ for $j \ge 2$. It will be necessary to construct $\mathscr{P}^{(j+1)}$ before constructing $\tilde{\mathcal{G}}^{(j+1)}$ as the partition ends up defining the hypergraph.

Construction of $\mathscr{P}^{(j+1)}$ and $\tilde{\mathcal{G}}^{(j+1)}$. We set the partition $\mathscr{P}^{(j+1)} = \mathscr{P}^{(j+1)}_{\text{new}} \cup \mathscr{P}^{(j+1)}_{\text{orig}}$ of $K^{(j+1)}_{\ell}(V_1,\dots,V_\ell)$ separately for each family $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ of $(j+1)$-tuples with $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\bar{\boldsymbol{b}})$.

Fix $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\bar{\boldsymbol{b}})$. We define the partition $\mathscr{P}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)}) \cup \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ of $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ by distinguishing three cases.

Case 1 ($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\bar{\boldsymbol{b}}) \setminus \hat{A}(j,\boldsymbol{b})$). Observe that $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ touches at least one of the exceptional graphs from the construction of $\mathscr{P}^{(2)}$. For the sake of consistency only (i.e., the partition $\mathscr{P}^{(j+1)}$ should contain a $(n/b_1,j+1,j+1)$-cylinder $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ for every $\boldsymbol{x}^{(j+1)} \in A(j+1,\bar{\boldsymbol{b}})$), we split $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ arbitrarily into b_{j+1} possibly empty classes. Clearly, all the $(n/b_1,j+1,j+1)$-cylinders $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ constructed in this way satisfy $\boldsymbol{x}^{(j+1)} \in \hat{A}(j+1,\bar{\boldsymbol{b}}) \setminus \hat{A}(j+1,\boldsymbol{b})$. The collection of these b_{j+1} disjoint $(n/b_1,j+1,j+1)$-cylinders defines $\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)})$. We set $\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \varnothing$.

Case 2 ($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$ and there exists $1 \le s \le j+1$ so that $\mathcal{P}^{(j)}(\partial_s \hat{\boldsymbol{x}}^{(j)}) \in \mathscr{P}^{(j)}_{\text{new}}$). We appeal to (*5*) of Fact 41 for j. Indeed, observe that $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ is $(\delta,1,r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ for any positive δ and integer r. Consequently, we may apply the Slicing Lemma, Lemma 31, with $\varrho = 1$, $p = \tilde{d}_{j+1}$, $\delta = 3\delta'_{j+1}/\tilde{d}_{j+1}$, and $r_{\text{SL}} = r'$ to $\mathcal{F}^{(j+1)} = \mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$. (Observe that $3\delta = 9\delta'/\tilde{d}_{j+1} < \tilde{d}_{j+1} = p\varrho$ by (44).) Since $1/p = 1/\tilde{d}_{j+1} = b_{j+1}$ by (49), we obtain a collection of $1/\tilde{d}_{j+1}$ pairwise edge-disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1},\tilde{d}_{j+1},r')$-regular $(n/b_1,j+1,j+1)$-cylinders $\mathcal{P}^{(j+1)}\big(\hat{\boldsymbol{x}}^{(j)},\alpha)\big)$ with $\alpha \in [b_{j+1}]$. Denote by

$$\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)}) = \Big\{\mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j)},\alpha)\big) \colon \ \alpha \in [b_{j+1}]\Big\}$$

the family of $(n/b_1,j+1,j+1)$-cylinders newly created. Again, set $\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \varnothing$. This concludes our treatment of Case 2.

Case 3 ($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$ and $\mathcal{P}^{(j)}(\partial_s \hat{\boldsymbol{x}}^{(j)}) \in \mathscr{P}^{(j)}_{\text{orig}}$ for every $1 \le s \le j+1$). By the assumption of this case and (I) of $(\mathcal{C}_j)$, we infer that there exists $\hat{\boldsymbol{y}}^{(j)} \in \hat{A}(j,\boldsymbol{a})$ such that $\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{y}}^{(j)}) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$. Recall the definition of $a^{\text{reg}}_{j+1} = a^{\text{reg}}_{j+1}(\hat{\boldsymbol{y}}^{(j)})$ (preceding (48)). Without loss of generality, let $\big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\big\}_{\alpha \in [a^{\text{reg}}_{j+1}]}$ be an enumeration of the $(\delta'_{j+1},\tilde{d}_{j+1},r')$-regular $(n/b_1,j+1,j+1)$-cylinders (regular w.r.t. $\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{y}}^{(j)}) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$). We set

$$\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \Big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\Big\}_{\alpha \in [a^{\text{reg}}_{j+1}]} \quad \text{and} \tag{59}$$

$$\mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \bigcup \Big\{\mathcal{P}^{(j+1)} \colon \ \mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})\Big\} = \bigcup_{\alpha \in [a^{\text{reg}}_{j+1}]} \mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\,.$$

Observe that $\mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ is $(a^{\text{reg}}_{j+1}\delta'_{j+1}, a^{\text{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ (cf. Proposition 47) and, as a consequence of (48), also $(3\delta'_{j+1}/\tilde{d}_{j+1}, a^{\text{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular. Then, $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big) \setminus \mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ is $(3\delta'_{j+1}/\tilde{d}_{j+1}, 1 - a^{\text{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular. We apply the Slicing Lemma to $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big) \setminus \mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ with $\varrho = 1 - a^{\text{reg}}_{j+1}\tilde{d}_{j+1}$, $p = \tilde{d}_{j+1}/\varrho$, $\delta = 3\delta'_{j+1}/\tilde{d}_{j+1}$ (yielding $3\delta < p\varrho$ by (44)) and $r_{\text{SL}} = r'$. Note that $1/p = \varrho/\tilde{d}_{j+1} = 1/\tilde{d}_{j+1} - a^{\text{reg}}_{j+1}$ is an integer by (47). We thus obtain collection $\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)})$ of $1/\tilde{d}_{j+1} - a^{\text{reg}}_{j+1}$ pairwise edge-disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1},\tilde{d}_{j+1},r')$-regular $(n/b_1,j+1,j+1)$-cylinders $\mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j)},\alpha)\big)$. Setting $\mathscr{P}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)}) \cup \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ yields a partition of $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ into $1/\tilde{d}_{j+1} - a^{\text{reg}}_{j+1} + a^{\text{reg}}_{j+1} = 1/\tilde{d}_{j+1} = b_{j+1}$ (by (49)) disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1},\tilde{d}_{j+1},r')$-regular $(n/b_1,j+1,j+1)$-cylinders. This concludes our treatment of Case 3.

Now, we set

$$\begin{aligned}\tilde{\mathcal{G}}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) &= \bigcup\Big\{\mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\rm orig}(\hat{\boldsymbol{x}}^{(j)}) \colon\ \mathcal{P}^{(j+1)} \subseteq \mathcal{G}^{(j+1)}\Big\}\\ \tilde{\mathcal{G}}^{(j+1)} &= \bigcup\Big\{\tilde{\mathcal{G}}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) \colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \bar{\boldsymbol{b}})\Big\}\end{aligned} \tag{60}$$

and we set

$$\mathscr{P}^{(j+1)}_{\rm new} = \bigcup\Big\{\mathscr{P}^{(j+1)}_{\rm new}(\hat{\boldsymbol{x}}^{(j)}) \colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \bar{\boldsymbol{b}})\Big\},$$

$$\mathscr{P}^{(j+1)}_{\rm orig} = \bigcup\Big\{\mathscr{P}^{(j+1)}_{\rm orig}(\hat{\boldsymbol{x}}^{(j)}) \colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \bar{\boldsymbol{b}})\Big\} \quad\text{and}\quad \mathscr{P}^{(j+1)} = \mathscr{P}^{(j+1)}_{\rm new} \cup \mathscr{P}^{(j+1)}_{\rm orig}.$$

It is left to verify (I)–(III) of statement $(\mathcal{C}_{j+1})$.

Confirmation of $(\mathcal{C}_{j+1})$. First we verify (I). To this end, we establish the equality of sets in (I) by decomposing the equality into its respective '$\subseteq$' and '$\supseteq$' parts, and begin by considering the former.

We verify the '$\subseteq$' component of the equality of the sets in (I) of $(\mathcal{C}_{j+1})$. Let $\mathcal{P}^{(j+1)} = \mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j+1)}, \alpha)\big) \in \mathscr{P}^{(j+1)}_{\rm orig}$. Owing to the construction of $\mathscr{P}^{(j+1)}$ above, $\mathcal{P}^{(j+1)}$ originates from Case 3. By the assumption of Case 3, we know that $\mathcal{P}^{(j)}(\partial_s\hat{\boldsymbol{x}}^{(j+1)}) \in \mathscr{P}^{(j)}_{\rm orig}$ for every $s \in [j+1]$. Consequently, from (I) of $(\mathcal{C}_j)$ we infer that for each $s \in [j+1]$, there exists $\boldsymbol{y}^{(j)}_s$ such that $\boldsymbol{\mathcal{R}}^{(j)}(\boldsymbol{y}^{(j)}_s) = \big\{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)}_s)\big\}_{h=1}^{j}$ is a $\big((\delta'_2, \dots, \delta'_j), (\tilde{d}_2, \dots, \tilde{d}_j), r'\big)$-regular $(n/a_1, j, j)$-complex and $\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s) = \mathcal{P}^{(j)}(\partial_s\hat{\boldsymbol{x}}^{(j+1)})$. Clearly,

$$\left\{\bigcup_{s\in[j+1]} \mathcal{R}^{(h)}(\boldsymbol{y}^{(j)}_s)\right\}_{h=1}^{j} \text{ is } \big((\delta'_2, \dots, \delta'_j), (\tilde{d}_2, \dots, \tilde{d}_j), r'\big)\text{-regular} \tag{61}$$

and $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)}) = \bigcup_{s\in[j+1]} \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s)$. Moreover, by the construction in Case 3 and $\mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\rm orig}$, there exists $\mathcal{R}^{(j+1)} \in \mathscr{R}^{(j+1)}$ such that $\mathcal{P}^{(j+1)} = \mathcal{R}^{(j+1)}$ and $\mathcal{R}^{(j+1)}$ is $(\delta'_{j+1}, \tilde{d}_{j+1}, r')$-regular with respect to $\bigcup_{s\in[j+1]} \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$. Then (61) yields the '$\subseteq$' component of the equality in (I) of $(\mathcal{C}_{j+1})$.

We now verify the '$\supseteq$' component of the equality in (I). To that end, let $\hat{\boldsymbol{y}}^{(j)} \in \hat{A}(j, \boldsymbol{a})$ and $\alpha \in [a_{j+1}]$ be given so that $\boldsymbol{\mathcal{R}}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)}, \alpha)\big) = \big\{\mathcal{R}^{(h)}\big((\boldsymbol{y}^{(j)}, \alpha)\big)\big\}_{h=1}^{j+1}$ is a $\big((\delta'_2, \dots, \delta'_{j+1}), (\tilde{d}_2, \dots, \tilde{d}_{j+1}), r'\big)$-regular complex. Hence, $\mathcal{R}^{(j+1)}(\partial_s\hat{\boldsymbol{y}}^{(j)}) \in \mathscr{P}^{(j)}_{\rm orig}$ for every $s \in [j+1]$ by the induction assumption (more precisely by (I) of $(\mathcal{C}_j)$). Moreover, the $(n, j+1, j+1)$-cylinder $\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)}, \alpha)\big)$ is $(\delta'_{j+1}, \tilde{d}_{j+1}, r')$-regular (i.e., $\alpha \in [a^{\rm reg}_{j+1}(\hat{\boldsymbol{x}}^{(j)})]$) and, consequently, $\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)}, \alpha)\big) \in \mathscr{P}^{(j+1)}_{\rm orig}$ (cf. (59) in Case 3). This concludes the proof of (I) of $(\mathcal{C}_j)$.

Since $j+1 \geq 3$, part (II) follows directly from (60) (recall, that we defined $\tilde{\mathcal{G}}^{(2)}$ slightly differently in (56) so that $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$).

In order to verify (III), we appeal to the induction assumption, and in particular, to (III) of $(\mathcal{C}_j)$. Observe that we only need to consider $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ for $\boldsymbol{x}^{(j+1)} \in A(j+1, \boldsymbol{b})$. Hence, it suffices to consider the constructions from Case 2 and Case 3. It is clear from the construction that in both of these cases we partitioned $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\boldsymbol{x}^{(j)})\big)$ into b_{j+1} different $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders. Consequently, (III) of $(\mathcal{C}_{j+1})$ holds and $(\mathcal{C}_{j+1})$ is verified.

This finishes the inductive proof of statement $(\mathcal{C}_i)$ for $2 \leq i \leq k-1$.

5.4. **Finale.** Having inductively defined partitions $\mathscr{P}^{(j)}$ and hypergraphs $\tilde{\mathcal{G}}^{(j)}$, $2 \leq j \leq k-1$, we proceed to construct the promised hypergraph $\tilde{\mathcal{G}}^{(k)}$ (see (62) below). Then we shall show that the conclusions of Lemma 30 hold for $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ and $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$.

Let $\hat{A}_{\rm reg}\big(\mathcal{P}^{(k-1)}_{\rm orig}, \mathcal{G}^{(k)}, k-1, \boldsymbol{b}\big)$ denote the set of $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$ for which $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{P}^{(k-1)}_{\rm orig}$ and $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. We set

$$\tilde{\mathcal{G}}^{(k)} = \bigcup\Big\{\mathcal{G}^{(k)} \cap \mathcal{K}^{(k-1)}_k\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) \colon\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}_{\rm reg}\big(\mathcal{P}^{(k-1)}_{\rm orig}, \mathcal{G}^{(k)}, k-1, \boldsymbol{b}\big)\Big\}. \tag{62}$$

It is left to verify that the earlier constructed family of partitions $\mathscr{P} = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ and $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$ satisfy the conclusion of Lemma 30.

Recall that for $2 \le j \le k-1$, we constructed $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ so that $(\mathcal{C}_j)$ and (57) holds. Consequently, by Fact 41 assertions (*1*)–(*5*) hold for every $j = 2, \dots, k-1$. The verification of Lemma 30 will rely on these assertions.

We first show that

$$\tilde{\boldsymbol{\mathcal{G}}} \text{ is an } (n, \ell, k)\text{-complex}. \tag{63}$$

By (*2*) of Fact 41 for $j = k-1$ we see that $\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k-1}$ is an $(n, \ell, k-1)$-complex. Now, let $K \in \tilde{\mathcal{G}}^{(k)}$. We have to show that $K \in \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})$. From (62), we infer that $K \in \mathcal{K}_k^{(k-1)}(\mathcal{G}^{(k-1)} \cap \mathcal{P}_{\text{orig}}^{(k-1)})$ and, consequently, by (II) of $(\mathcal{C}_{k-1})$, we have $K \in \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})$. Therefore, $\tilde{\mathcal{G}}^{(k-1)}$ underlies $\tilde{\mathcal{G}}^{(k)}$ and (63) follows.

Now we show that

$$\tilde{\boldsymbol{d}} \text{ is componentwise bigger than } \tilde{\boldsymbol{c}}. \tag{64}$$

Suppose $\tilde{d}_j \le \tilde{c}_j$ for some $2 \le j \le k-1$. Recall, that $\tilde{\boldsymbol{d}}$ was given by Corollary 28 as the density vector of $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$. Moreover, $L_k \ge |A(k-1, \boldsymbol{a})|$ and hence $|\hat{A}(j-1, \boldsymbol{a})| < 2^\ell L_k^k$ for $j = 2, \dots, k$. Therefore, the assumption $\tilde{d}_j \le \tilde{c}_j = 1/(2^{\ell+1} L_k^k)$ (see (46)) implies that the number of j-tuples in $(\delta'_j, \tilde{d}_j, r')$-regular polyads of $\mathscr{R}$ is at most $2^\ell L_k^k(\tilde{d}_j + \delta'_j)n^j \le 2^{\ell+1} L_k^k \tilde{c}_j n^j = n^j/2$. On the other hand, by (50), all but at most $\mu\binom{k}{j}n^j$ crossing j-tuples belong to $(\delta'_j, \tilde{d}_j, r')$-regular polyads of $\mathscr{R}$. Since $(1/2 + \mu\binom{k}{j})n^j \le \binom{\ell}{j}n^j$ the assumption $\tilde{d}_j \le \tilde{c}_j$ must be wrong and we infer that $\tilde{d}_j > \tilde{c}_j$ for every $2 \le j \le k-1$, as claimed in (64).

Summarizing the above we infer that $\tilde{d}_j > \tilde{c}_j$ for every $2 \le j \le k-1$, as claimed in (64).

Using (III) of $(\mathcal{C}_{k-1})$ combined with (44) and (45) yields that

$$\mathscr{P} = \mathscr{P}_{k-1} \text{ is an almost perfect } (\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})\text{-family of partitions}. \tag{65}$$

Moreover, (*3*) of Fact 41 for $j = k-1$ states that

$$\mathscr{P} = \mathscr{P}_{k-1} \text{ refines } \tilde{\boldsymbol{\mathcal{G}}}. \tag{66}$$

From (63)–(66) we infer that it is left to show (*i*)–(*v*) of Lemma 30, only. We observe that (*i*) is immediate from the construction of $\tilde{\mathcal{G}}^{(k)}$ in (62). Also, due to (62), (II) of $(\mathcal{C}_j)$ for $j = 2, \dots, k-1$ (see also (*1*) of Fact 41), and the definition of $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$ we have (*ii*) of Lemma 30.

Now we verify (*iii*) of Lemma 30. For $3 \le j < k$ it is given by part (*4*) of Fact 41. For $j = k$, we recall the definition of $\tilde{\mathcal{G}}^{(k)}$ in (62) and consider $\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}$. There are two reasons for a k-tuple $K \in \mathcal{G}^{(k)}$ to be in $\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}$. Either $K \notin \mathcal{K}_k^{(k-1)}(\mathcal{P}_{\text{orig}}^{(k-1)})$ or K belongs to a polyad $\hat{\mathcal{P}}^{(k-1)}$ such that $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.

Consider a k-tuple of the first type, i.e., $K \notin \mathcal{K}_k^{(k-1)}(\mathcal{P}_{\text{orig}}^{(k-1)})$. Owing to (I) of $(\mathcal{C}_{k-1})$ we see that K belongs to a $\big((\delta'_2, \dots, \delta'_{k-1}), (\tilde{d}_2, \dots, \tilde{d}_{k-1}), r'\big)$-irregular $(n/a_1, k, k-1)$-complex of the original family of partitions $\mathscr{R}$. Consequently, by (50) (with $j = k$) there are at most μn^k k-tuples K of the first type ($K \notin \mathcal{K}_k^{(k-1)}(\mathcal{P}_{\text{orig}}^{(k-1)})$).

Now consider a k-tuple K, which is not of the first type, but of the second type. In particular, $K \in \mathcal{K}_k^{(k-1)}(\mathcal{P}_{\text{orig}}^{(k-1)})$ and $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, the underlying polyad of K in the family of partitions $\mathscr{P}$. From (I) of $(\mathcal{C}_{k-1})$ we infer that $\hat{\mathcal{P}}^{(k-1)}$ corresponds to k different $(n, k-1, k-1)$-cylinders, which are all elements of $\mathscr{R}^{(k-1)}$. Since $\mathscr{R}$ is a (δ'_k, r')-regular partition w.r.t. $\mathcal{G}^{(k)}$ and $\tilde{\delta}_k \ge \delta'_k$ and $\tilde{r} \le r'$ (cf. (44) and (45)), there are at most $\delta'_k n^k$ k-tuples $K \in \mathcal{G}^{(k)} \cap \mathcal{K}_k^{(k-1)}(\mathcal{P}_{\text{orig}}^{(k-1)})$ so that $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. to the underlying polyad $\hat{\mathcal{P}}^{(k-1)}$ of K.

Summarizing the above, we infer that

$$|\mathcal{G}^{(k)} \triangle \tilde{\mathcal{G}}^{(k)}| = |\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}| \le (\mu + \delta'_k)n^k \overset{(43)}{=} 2\mu n^k.$$

Consequently, by the choice of δ'_k in (43), the following holds for every $k \le i \le \ell$,

$$\left|\mathcal{K}_i^{(k)}(\tilde{\mathcal{G}}^{(k)}) \triangle \mathcal{K}_i^{(k)}(\mathcal{G}^{(k)})\right| \le 2\mu n^k \times \binom{\ell - k}{i - k} n^{i-k} \overset{(43)}{\le} \tilde{\delta}_k \prod_{h=2}^{k} d_h^{\binom{i}{h}} \times n^i,$$

which completes the verification of (*iii*) of Lemma 30.

We further note that (*iv*) of Lemma 30 is an immediate consequence of $b_1 = a_1 \le \operatorname{rank} \mathscr{R} \le L_k \le \tilde{L}_k$ (cf. (46)), $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ and the assumption of Lemma 30 that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, 1)$-regular complex.

Finally, we show (v) of Lemma 30 as follows:

$$\operatorname{rank}\mathscr{P} = |A(k-1,\bar{\boldsymbol{b}})| = \binom{\ell}{k-1} b_1^{k-1}(b_2+1)^{\binom{k-1}{2}} \prod_{j=3}^{k-1} b_j^{\binom{k-1}{j}}$$

$$\leq \binom{\ell}{k-1} a_1^{k-1}(2b_2)^{\binom{k-1}{2}} \prod_{j=3}^{k-1} b_j^{\binom{k-1}{j}} \overset{(49)}{\leq} 2^{\ell+2\binom{k-1}{2}} L_k^{k-1} \prod_{j=2}^{k-1} \left(\frac{1}{\tilde{d}_j}\right)^{\binom{k-1}{j}} .$$

Then (v) follows from $\tilde{\boldsymbol{c}} \leq \tilde{\boldsymbol{d}}$ and the choice of $\tilde{L}_k$ in (46).

This completes the proof of Lemma 30.

6. Proofs concerning Cleaning Phase II

We prove Lemma 37 and Lemma 38 in this section. We work in the context of Setup 36, the environment after Cleaning Phase I (after an application of Lemma 30) with the constants from Figure 2. The main objective of this section is to construct the complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ stated in Lemma 37 and Lemma 38. We prove these lemmas in Section 6.2 and Section 6.3, respectively. The following section, Section 6.1, contains some preliminary facts, which are immediate consequences of the choice of constants given in Section 4.4.1 (see Figure 2).

6.1. Preliminary Facts.

We start with the following facts which we apply liberally in the remainder of this section. The first two facts are immediate consequences of $\mathbf{IHC}_{k-1,\ell}$ and the choice of constants in Section 4.4.1 (applied to differing setups).

Fact 42. *For all integers $2 \leq j < k$ and $j < i \leq \ell$ and every $\Lambda_i \in \binom{[\ell]}{i}$,*

$$\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| = (1\pm\eta)\prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i , \tag{67}$$

$$\left|\mathcal{K}_i^{(j)}\big(\tilde{\mathcal{G}}^{(j)}[\Lambda_i]\big)\right| = \big(1\pm(\eta+\tilde{\delta}_k)\big)\prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i . \tag{68}$$

Consequently, by the choice of η in (19) and $\tilde{\delta}_k \leq 1/8$ in (22),

$$\left|\mathcal{K}_i^{(j)}\big(\tilde{\mathcal{G}}^{(j)}[\Lambda_i]\big)\right| \geq \frac{1-1/4-\tilde{\delta}_k}{1+1/4}\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| \geq \frac{1}{2}\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| . \tag{69}$$

Proof. Due to the choice of $\boldsymbol{\delta} = (\delta_2,\dots,\delta_{k-1})$ and r (cf. (20), (26), and (27)) for $2 \leq j < k$, the complex $\boldsymbol{\mathcal{G}^{(j)}} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ satisfies the assumption of $\mathbf{IHC}_{k-1,\ell}$. As such, we conclude that (67) holds. Since $\tilde{\boldsymbol{\mathcal{G}}}$ is given by Lemma 30, it satisfies (iii) of that lemma and (68) follows. □

In the following fact, $\check{\boldsymbol{\mathcal{H}}}^{\boldsymbol{(j-1)}}$ represents an arbitrary regular $(n/b_1, i, j-1)$-complex arising from an application of Lemma 30 (i.e., the complex $\check{\boldsymbol{\mathcal{H}}}^{\boldsymbol{(j-1)}}$ is "built from blocks" of the partition $\mathscr{P}$).

Fact 43. *If $1 \leq j-1 < k$ and $j \leq i \leq \ell$ and $\check{\boldsymbol{\mathcal{H}}}^{\boldsymbol{(j-1)}} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ is a $\big((\tilde{\delta}_2,\dots,\tilde{\delta}_{j-1}),(\tilde{d}_2,\dots,\tilde{d}_{j-1}),\tilde{r}\big)$-regular $(n/b_1, i, j-1)$-complex, then*

$$\left|\mathcal{K}_i^{(j-1)}\big(\check{\mathcal{H}}^{(j-1)}\big)\right| = (1\pm\tilde{\eta})\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}} \times \left(\frac{n}{b_1}\right)^i . \tag{70}$$

In particular, for every $1 \leq j-1 < k$ and every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1,\boldsymbol{b})$,

$$\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\right| = (1\pm\tilde{\eta})\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}} \times \left(\frac{n}{b_1}\right)^j . \tag{71}$$

Proof. Similarly as in the proof of Fact 42, by the choice of $\tilde{\eta}$ and $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \dots, \tilde{\delta}_{k-1})$ and $\tilde{r}$ (cf. (22) and (24)), we infer that $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j-1})}$ for $2 \le j-1 \le k-1$ satisfies the assumptions of $\mathbf{IHC}_{k-1,\ell}$ and, consequently, (70) of Fact 43 holds. □

Recall that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex where by (ii) and (iii) of Lemma 30 (with $i = j$)

$$\mathcal{G}^{(1)} = \tilde{\mathcal{G}}^{(1)},\ \mathcal{G}^{(2)} = \tilde{\mathcal{G}}^{(2)} \text{ and } \left|\mathcal{G}^{(j)} \setminus \tilde{\mathcal{G}}^{(j)}\right| \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \text{ for } 3 \le j \le k\,. \tag{72}$$

Since $\tilde{\delta}_k$ is significantly smaller than δ_j, $3 \le j \le k$ (cf. Figure 2), we infer the following fact by a standard argument.

Fact 44. *The (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}}$ is $(2\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular.*

Proof. By the choice of the constants in Section 4.4.1, we infer that $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ (see Lemma 30 (ii)) and hence $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$.

We now show that $\tilde{\mathcal{G}}^{(j)}$ is $(2\delta_j, d_j, r)$-regular w.r.t. $\tilde{\mathcal{G}}^{(j-1)}$ for each $j \ge 3$. Let j and $\Lambda_j \in \binom{[\ell]}{j}$ be fixed. Let $\boldsymbol{\mathcal{Q}}^{(\boldsymbol{j-1})} = \left\{\mathcal{Q}_s^{(j-1)}\right\}_{s\in[r]}$ be a family of subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j] \subseteq \mathcal{G}^{(j-1)}[\Lambda_j]$ such that

$$\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \ge 2\delta_j \left|\mathcal{K}_j^{(j-1)}\left(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\right)\right|.$$

From (67) and (69), we then infer

$$\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| > \delta_j \left|\mathcal{K}_j^{(j-1)}\left(\mathcal{G}^{(j-1)}[\Lambda_j]\right)\right| \ge \delta_j(1-\eta) \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j. \tag{73}$$

Since $\boldsymbol{\mathcal{Q}}^{(\boldsymbol{j-1})}$ is a family of subhypergraphs of $\mathcal{G}^{(j-1)}[\Lambda_j]$ and since $\mathcal{G}^{(j)}[\Lambda_j]$ is (δ_j, d_j, r)-regular with respect to $\mathcal{G}^{(j-1)}[\Lambda_j]$, we see

$$\left|\mathcal{G}^{(j)}[\Lambda_j] \cap \bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| = (d_j \pm \delta_j) \left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|. \tag{74}$$

On the other hand, (72) and (74) imply

$$\begin{aligned}\left|\tilde{\mathcal{G}}^{(j)}[\Lambda_j] \cap \bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| &\overset{(72)}{=} \left|\mathcal{G}^{(j)}[\Lambda_j] \cap \bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \pm \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \\ &\overset{(74)}{=} (d_j \pm \delta_j) \left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \pm \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \\ &= (d_j \pm 2\delta_j) \left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|,\end{aligned}$$

where the last equality uses (73) and $\tilde{\delta}_k d_j \le \delta_j^2(1-\eta)$ for $j \ge 3$. □

6.2. **Proof of Lemma 37.** The proof of Lemma 37 will take place in stages. Setting $\mathcal{H}^{(1)} = \tilde{\mathcal{G}}^{(1)}$ and $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)}$ satisfies part (a) of Lemma 37. We prove part (b) of Lemma 37 in Section 6.2.1 and part (c) in Section 6.2.3.

6.2.1. *Proof of Property (b) of Lemma 37.* We prove part (b) by induction on j.

Induction Start. Recall that we set $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)}$. Consequently, the symmetric difference considered in part $(b2)$ of Lemma 37 is empty. Hence, $(b2)$ holds trivially for $j = 2$ and it is left to verify $(b1)$. To that end, let $\hat{\boldsymbol{x}}^{(1)} = ((\lambda_1, \lambda_2), (\beta_1, \beta_2)) \in \hat{A}(\mathcal{H}^{(1)}, 1, \boldsymbol{b}) = \hat{A}(1, \boldsymbol{b})$ be fixed. From part (iv) of Lemma 30, we infer $d\big(\tilde{\mathcal{G}}^{(2)}\big|\hat{\mathcal{P}}^{(1)}\big(\hat{\boldsymbol{x}}^{(1)}\big)\big) = d_2 \pm \tilde{L}_k^2\delta_2$. From (ii) of Definition 29, we then infer

$$\frac{d_2 - \tilde{L}_k^2\delta_2}{\tilde{d}_2 + \tilde{\delta}_2} \le \Big|I\big(\hat{\boldsymbol{x}}^{(1)}\big)\Big| \le \frac{d_2 + \tilde{L}_k^2\delta_2}{\tilde{d}_2 - \tilde{\delta}_2}\,.$$

As such, to verify $(b1)$, we may show that the left-hand side of the last inequality is bigger than $d_2b_2 - 1$ and the right-hand side is less than $d_2b_2 + 1$. Consequently, it suffices to verify

$$(\tilde{d}_2 + \tilde{\delta}_2)(d_2b_2 - 1) < d_2 - \tilde{L}_k^2\delta_2 \qquad \text{and} \qquad d_2 + \tilde{L}_k^2\delta_2 < (d_2b_2 + 1)(\tilde{d}_2 - \tilde{\delta}_2)\,. \tag{75}$$

The proofs of both inequalities are similar and we only present the details for the first one here.

We consider the left-hand side of the first inequality in (75) and see

$$(\tilde{d}_2 + \tilde{\delta}_2)(d_2b_2 - 1) < \tilde{d}_2d_2b_2 - \tilde{d}_2 + \tilde{\delta}_2d_2b_2 \le d_2(1 + \tilde{\delta}_2/\tilde{d}_2) - \tilde{d}_2 + \tilde{\delta}_2b_2 \le d_2 + \tilde{\delta}_2/\tilde{d}_2 - \tilde{d}_2 + \tilde{\delta}_2b_2\,. \tag{76}$$

where we use (i) of Definition 29 for the last inequality. Again, from (i) of Definition 29 and $\tilde{d}_2 > \tilde{\delta}_2$, we know $b_2 < 2/\tilde{d}_2$. Therefore, using $\tilde{\delta}_2 \lll \tilde{d}_2$ gives the following bound for the right-hand side of (76)

$$d_2 + \tilde{\delta}_2/\tilde{d}_2 - \tilde{d}_2 + \tilde{\delta}_2b_2 < d_2 - \tilde{d}_2 + 3\tilde{\delta}_2/\tilde{d}_2 < d_2 - \tilde{d}_2 + \sqrt{\tilde{\delta}_2}\,. \tag{77}$$

Summarizing (76) and (77), the first inequality of (75) follows from the choice of constants $\tilde{d}_2 \ggg \tilde{\delta}_2 \ggg \tilde{L}_k^2\delta_2$ (see Figure 2), by

$$(\tilde{d}_2 + \tilde{\delta}_2)(d_2b_2 - 1) < d_2 - \tilde{d}_2 + \sqrt{\tilde{\delta}_2} < d_2 - \tilde{L}_k^2\delta_2\,.$$

Induction Step. Assume that for $2 \le j < k$, part (b) of Lemma 37 holds for $j-1$ with inductively defined complex $\boldsymbol{\mathcal{H}^{(j-1)}} = \{\mathcal{H}^{(h)}\}_{h=1}^{j-1}$. We construct the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, and hypergraph $\mathcal{H}^{(j)}$ satisfying $(b1)$ and $(b2)$. We first define the following set of indices crucial for our constructions.

For a vector $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, set

$$J\big(\hat{\boldsymbol{x}}^{(j-1)}\big) = \Big\{\beta \in [b_j] \colon \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big) \subseteq \tilde{\mathcal{G}}^{(j)}\Big\}\,. \tag{78}$$

For $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, observe

$$\Big|\tilde{\mathcal{G}}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\mathcal{P}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\Big| = \sum_{\beta \in J(\hat{\boldsymbol{x}}^{(j-1)})} \Big|\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big)\Big|\,. \tag{79}$$

Now we construct the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$ for every $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$.

- *If* $\big|J(\hat{\boldsymbol{x}}^{(j-1)})\big| > d_jb_j$, *then* $I(\hat{\boldsymbol{x}}^{(j-1)})$ *is defined by removing* $\big|J(\hat{\boldsymbol{x}}^{(j-1)})\big| - d_jb_j$ *arbitrary indices from* $J(\hat{\boldsymbol{x}}^{(j-1)})$.
- *If* $\big|J(\hat{\boldsymbol{x}}^{(j-1)})\big| < d_jb_j$, *then* $I(\hat{\boldsymbol{x}}^{(j-1)})$ *is defined by adding* $d_jb_j - \big|J(\hat{\boldsymbol{x}}^{(j-1)})\big|$ *arbitrary indices of* $[b_j] \setminus J(\hat{\boldsymbol{x}}^{(j-1)})$ *to* $J(\hat{\boldsymbol{x}}^{(j-1)})$.

This defines the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$.

For upcoming considerations, we define the set $\hat{B}^{(j-1)}$ of addresses $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ for which $|J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})|$ is 'too big'. More precisely, we define

$$\hat{B}^{(j-1)} = \Big\{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \colon \big|J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})\big| > \sqrt{\delta_j}d_jb_j\Big\}. \tag{80}$$

We prove the following claim in Section 6.2.2.

Claim 45. $\Big|\hat{B}^{(j-1)}\Big| < 2\sqrt{\delta_j}\prod_{h=2}^{j-1}(d_hb_h)^{\binom{j}{h}} \times b_1^j\,.$

We define hypergraph $\mathcal{H}^{(j)}$ as

$$\mathcal{H}^{(j)} = \bigcup\Big\{\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \wedge \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})\Big\}\,. \tag{81}$$

We now prove Property (b) of Lemma 37, and to that end, we establish both parts $(b1)$ and $(b2)$. Note, however, that with $I(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, and $\mathcal{H}^{(j)}$ constructed above, Property $(b1)$ of Lemma 37 follows immediately. Thus, it remains to prove Property $(b2)$.

Let $j \le i \le \ell$ be fixed and consider the set $\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$. Clearly, for every i-tuple $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$, there exists a j-tuple $J_0 \in \binom{I_0}{j}$ such that $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$. We note that one possibility for $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$ is that $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. Since we have some control over the cardinality of $\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$ (by the induction assumption on $(b1)$), it is natural to split the i-tuples $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$ into two parts, $\mathfrak{K}_i^{(j)}(1)$ and $\mathfrak{K}_i^{(j)}(2)$, depending on whether there is a $J_0 \in \binom{I_0}{j}$ as described above. More precisely, we define

$$\mathfrak{K}_i^{(j)}(1) = \left\{ I_0 \in \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \right) \colon \exists\, J_0 \in \binom{I_0}{j} \text{ so that } J_0 \in \left(\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)}) \right) \right\}$$

and

$$\begin{aligned} \mathfrak{K}_i^{(j)}(2) &= \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \right) \Big\backslash \mathfrak{K}_i^{(j)}(1) \\ &= \left\{ I_0 \in \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \right) \colon \forall\, J_0 \in \binom{I_0}{j} \quad J_0 \notin \left(\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)}) \right) \right\}. \end{aligned}$$

Observe that we may rewrite $\mathfrak{K}_i^{(j)}(2)$ as

$$\mathfrak{K}_i^{(j)}(2) = \left\{ I_0 \in \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \right) \colon \forall\, J_0 \in \binom{I_0}{j} \quad J_0 \in \left(\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \cap \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)}) \right) \right\}. \tag{82}$$

Indeed, for the equality (of sets) in (82), the inclusion '$\supseteq$' is obvious. The opposite inclusion '$\subseteq$' follows from the fact that $\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \subseteq \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \cup \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$ and, consequently, for every considered I_0 and $J_0 \in \binom{I_0}{j}$, we have $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \cup \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. The '$\subseteq$' inclusion then follows. Note that from (82), we infer

$$\mathfrak{K}_i^{(j)}(2) \subseteq \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)}) \cap \mathcal{K}_i^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)}). \tag{83}$$

We now consider a subdivision of $\mathfrak{K}_i^{(j)}(2)$. From (82), we infer that all $I_0 \in \mathfrak{K}_i^{(j)}(2)$ only 'touch' polyads $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. Let $\mathfrak{K}_i^{(j)}(2,1)$ be the set of all $I_0 \in \mathfrak{K}_i^{(j)}(2)$ which 'touch' a bad polyad $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ (bad in the sense of Claim 45) with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}^{(j-1)} \subseteq \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. Formally, set

$$\mathfrak{K}_i^{(j)}(2,1) = \left\{ I_0 \in \mathfrak{K}_i^{(j)}(2) \colon \exists\ J_0 \in \binom{I_0}{j} \text{ and } \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}^{(j-1)} \text{so that } J_0 \in \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \right\}.$$

The remaining $I_0 \in \mathfrak{K}_i^{(j)}(2) \setminus \mathfrak{K}_i^{(j)}(2,1)$ 'touch' only good polyads. However, as observed earlier, for every such I_0, there exists a $J_0 \in \binom{I_0}{j}$ such that $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$. Recall that the union of the sets $J(\hat{\boldsymbol{x}}^{(j-1)})$ with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ represents $\tilde{\mathcal{G}}^{(j)} \cap \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)})$ (cf. (78)) and similarly the union of $I(\hat{\boldsymbol{x}}^{(j-1)})$ represents $\mathcal{H}^{(j)}$ (cf. (81)). Consequently, $J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ represents the difference of $\tilde{\mathcal{G}}^{(j)}$ and $\mathcal{H}^{(j)}$ on the underlying polyad having address $\hat{\boldsymbol{x}}^{(j-1)}$. Hence, we infer that for every $I_0 \in \mathfrak{K}_i^{(j)}(2) \setminus \mathfrak{K}_i^{(j)}(2,1)$, there exist a $J_0 \in \binom{I_0}{j}$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)}$ and $\alpha \in J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ so that $J_0 \in \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$. We therefore set

$$\begin{aligned} \mathfrak{K}_i^{(j)}(2,2) = \Bigg\{ I_0 \in \mathfrak{K}_i^{(j)}(2) \colon \exists\ J_0 \in \binom{I_0}{j},\ & \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} \\ & \text{and } \alpha \in J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)}) \text{ so that } J_0 \in \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) \Bigg\}. \end{aligned}$$

Note that $\mathfrak{K}_i^{(j)}(2,1)$ and $\mathfrak{K}_i^{(j)}(2,2)$ are not necessarily disjoint. However, $\mathfrak{K}_i^{(j)}(2) = \mathfrak{K}_i^{(j)}(2,1) \cup \mathfrak{K}_i^{(j)}(2,2)$ and therefore

$$\left| \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \right| \le \left| \mathfrak{K}_i^{(j)}(1) \right| + \left| \mathfrak{K}_i^{(j)}(2,1) \right| + \left| \mathfrak{K}_i^{(j)}(2,1) \right|. \tag{84}$$

In what follows, we derive an upper bound for each term of the right-hand side of (84) which all combined yield part $(b2)$ of Lemma 37.

Bounding $|\mathfrak{K}_i^{(j)}(1)|$. The upper bound on $|\mathfrak{K}_i^{(j)}(1)|$ follows from the induction assumption on part (b) of Lemma 37. First, observe that

$$\mathfrak{K}_i^{(j)}(1) \subseteq \mathcal{K}_i^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle\, \mathcal{K}_i^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\,. \tag{85}$$

Indeed, if $I_0 \in \mathfrak{K}_i^{(j)}(1)$, then (immediately) $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle\, \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$. Assume, without loss of generality, that $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \setminus \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$ (the other case is symmetric). Since, $\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \subseteq \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)})$ we have

$$I_0 \in \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)})\,. \tag{86}$$

On the other hand, due to the definition of $\mathfrak{K}_i^{(j)}(1)$, for each such I_0 there exists $J_0 \in \binom{I_0}{j}$ satisfying $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle\, \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. From (86), we also have $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)})$ and hence $J_0 \notin \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. Consequently, $I_0 \notin \mathcal{K}_i^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$ and thus $I_0 \in \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle\, \mathcal{K}_i^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$ which yields (85).

Now the induction assumption on $(b2)$, with j replaced by $j-1$, gives the following:

$$\left|\mathfrak{K}_i^{(j)}(1)\right| \le \left|\mathcal{K}_i^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle\, \mathcal{K}_i^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\right| \le \delta_{j-1}^{1/3} \left(\prod_{h=2}^{j-1} d_h^{\binom{i}{h}}\right) n^i \le \frac{1}{3}\delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i \tag{87}$$

where the last inequality follows from the choice of constants summarized in Figure 2 ensuring $\delta_{j-1}^{1/3} \lll \delta_j^{1/3} d_j^{\binom{i}{j}}$.

Bounding $|\mathfrak{K}_i^{(j)}(2,1)|$. By Property $(b1)$, there are $\binom{\ell-j}{i-j}\big(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\big) b_1^{i-j}$ different ways to complete any given $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ to a $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \ldots, \tilde{d}_{j-1}), \tilde{r}\big)$-regular $(n/b_1, i, j-1)$-subcomplex $\check{\boldsymbol{\mathcal{H}}}^{\boldsymbol{(j-1)}} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ of $\boldsymbol{\mathcal{H}}^{\boldsymbol{(j-1)}}$. Then, (70) of Fact 43 yields that for each such $\check{\mathcal{H}}^{(j-1)}$,

$$\left|\mathcal{K}_i^{(j-1)}\big(\check{\mathcal{H}}^{(j-1)}\big)\right| \le (1+\tilde{\eta}) \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i \le 2 \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i.$$

Using (83) and Claim 45 (for the second inequality below), we therefore see

$$\begin{aligned}\left|\mathfrak{K}_i^{(j)}(2,1)\right| &\le \left|\hat{B}^{(j-1)}\right| \times \binom{\ell-j}{i-j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) b_1^{i-j} \times 2 \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i \\ &\le 4\binom{\ell-j}{i-j}\sqrt{\delta_j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{i}{h}}\right) \left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}}\right) n^i \le 4\binom{\ell-j}{i-j}\sqrt{\delta_j} \left(\prod_{h=2}^{j-1} (d_h b_h \tilde{d}_h)^{\binom{i}{h}}\right) n^i,\end{aligned}$$

and so, by (31) and the choice of $\delta_j \lll d_j$, we have the upper bound

$$\left|\mathfrak{K}_i^{(j)}(2,1)\right| \le 4\binom{\ell-j}{i-j}(1+\nu)\sqrt{\delta_j} \left(\prod_{h=2}^{j-1} d_h^{\binom{i}{h}}\right) n^i \le \frac{1}{3}\delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i\,. \tag{88}$$

Bounding $|\mathfrak{K}_i^{(j)}(2,2)|$. First, let $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)}$ and $\alpha \in J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle\, I(\hat{\boldsymbol{x}}^{(j-1)})$ be fixed and consider the $(n/b_1, j, j)$-complex implicitly given by $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$. By Property $(b1)$, there are

$$\binom{\ell-j}{i-j} b_1^{i-j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) b_j^{\binom{i}{j}-1} = d_j^{1-\binom{i}{j}} \binom{\ell-j}{i-j} b_1^{i-j} \prod_{h=2}^{j} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}$$

ways to complete any $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ to a $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_j), (\tilde{d}_2, \ldots, \tilde{d}_j), \tilde{r}\big)$-regular $(n/b_1, i, j)$-complex $\check{\boldsymbol{\mathcal{H}}}^{\boldsymbol{(j)}} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j}$ in such a way that $\{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ is a subcomplex of $\boldsymbol{\mathcal{H}}^{\boldsymbol{(j-1)}}$.

Then (70) of Fact 43 yields

$$\left|\mathcal{K}_i^{(j)}(\check{\mathcal{H}}^{(j)})\right| \le (1+\tilde{\eta}) \prod_{h=2}^{j} \tilde{d}_h^{\binom{i}{h}} \times \left(\frac{n}{b_1}\right)^i$$

for every such $\check{\mathcal{H}}^{(j)}$. Now, summing over all choices $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} \subseteq \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ and $\alpha \in J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ gives

$$\begin{aligned} \left|\mathfrak{K}_i^{(j)}(2,2)\right| &\overset{(83)}{\le} \left|\hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\right| \times \max\left\{ \left|J\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \triangle I\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| : \hat{\boldsymbol{x}}^{(j-1)} \notin \hat{B}^{(j-1)} \right\} \times \\ &\times\ d_j^{1-\binom{i}{j}} \binom{\ell-j}{i-j} b_1^{i-j} \left(\prod_{h=2}^{j} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) \times (1+\tilde{\eta}) \left(\prod_{h=2}^{j} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i . \end{aligned} \tag{89}$$

By Property (*b1*),

$$\left|\hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\right| = \binom{\ell}{j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}}\right) b_1^j .$$

Also note that for each $\hat{\boldsymbol{x}}^{(j-1)} \notin \hat{B}^{(j-1)}$, $\left|J(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})\right| \le \sqrt{\delta_j} d_j b_j$. Consequently, the right-hand side of (89) is less than

$$d_j^{1-\binom{i}{j}} \binom{\ell}{j} \binom{\ell-j}{i-j} \sqrt{\delta_j} (1+\tilde{\eta}) \left(\prod_{h=2}^{j} (d_h b_h \tilde{d}_h)^{\binom{i}{h}}\right) n^i$$

Now, using (31), the choice of $\tilde{\eta}$ and $\delta_j \lll d_j$ yields

$$\left|\mathfrak{K}_i^{(j)}(2,2)\right| \le \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i . \tag{90}$$

Finally, (84) combined with (87), (88), and (90) yields part (*b2*) of Lemma 37. In order to complete the proof of part (*b*) of Lemma 37 we still have to verify Claim 45.

6.2.2. *Proof of Claim 45.* The proof is rather straightforward in the genre of hypergraph regularity. We first split the set $\hat{B}^{(j-1)}$ into two parts $\hat{B}_+^{(j-1)}$ and $\hat{B}_-^{(j-1)}$ as follows:

$$\begin{aligned} \hat{B}_+^{(j-1)} &= \left\{ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) : \left|J\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| > \left(1+\sqrt{\delta_j}\right) d_j b_j \right\} \\ \hat{B}_-^{(j-1)} &= \left\{ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) : \left|J\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| < \left(1-\sqrt{\delta_j}\right) d_j b_j \right\} . \end{aligned} \tag{91}$$

We prove the following claim which in view of Fact 44 is a slightly stronger statement than Claim 45.

Claim 45′. *If for some* $* \in \{+,-\}$,

$$\left|\hat{B}_*^{(j-1)}\right| \ge \sqrt{\delta_j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}}\right) b_1^j ,$$

then $\tilde{\mathcal{G}}^{(j)}$ *is not* $(2\delta_j, d_j, r)$*-regular w.r.t.* $\tilde{\mathcal{G}}^{(j-1)}$.

Proof. We prove the case $* = -$ only with the other case very similar. We assume there exists an ordered set $\Lambda_j \in \binom{[\ell]}{j}_<$ such that

$$\left|\hat{B}_-^{(j-1)}[\Lambda_j]\right| \ge \frac{\sqrt{\delta_j}}{\binom{\ell}{j}} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}}\right) b_1^j \tag{92}$$

where $\hat{B}_-^{(j-1)}[\Lambda_j]$ is the set of $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \dots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{B}_-^{(j-1)}$ such that $\hat{\boldsymbol{x}}_0 = \Lambda_j$.

We show that (92) implies that $\tilde{\mathcal{G}}^{(j)}$ is irregular. Note that the polyad addresses $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{B}_-^{(j-1)}[\Lambda_j]$ considered in (92) correspond to subhypergraphs of $\mathcal{H}^{(j-1)}$ and not necessarily to subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}$.

The set $\hat{\Gamma}_-^{(j-1)}[\Lambda_j]$ which we define below is the subset of those polyad addresses of $\hat{B}_-^{(j-1)}[\Lambda_j]$ which correspond to subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}$ as well. Only those addresses are useful to verify Claim 45′. We therefore set

$$\hat{\Gamma}_-^{(j-1)}[\Lambda_j] = \hat{B}_-^{(j-1)}[\Lambda_j] \cap \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big) \tag{93}$$

and

$$\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)} = \Big\{\hat{\mathcal{P}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\} = \Big\{\hat{\mathcal{Q}}_1^{(j-1)}, \dots, \hat{\mathcal{Q}}_t^{(j-1)}\Big\}$$

where $t = \big|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\big|$. In what follows, we show

$$\bigg|\bigcup_{s\in[t]} \Big\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big\}\bigg| > 2\delta_j \left|\mathcal{K}_j^{(j-1)}\left(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\right)\right| \tag{94}$$

and

$$d\left(\tilde{\mathcal{G}}^{(j)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j - 2\delta_j\,. \tag{95}$$

From (29), we see that $r \geq \big|\hat{A}(j-1,\boldsymbol{b})\big| \geq t$. Therefore, establishing (94) and (95) proves Claim 45′.

We first verify (94). Observe that due to the definition of $\hat{\Gamma}_-^{(j-1)}[\Lambda_j]$ in (93),

$$\begin{aligned}\hat{B}_-^{(j-1)}[\Lambda_j] \setminus \hat{\Gamma}_-^{(j-1)}[\Lambda_j] &\subseteq \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\Big\backslash \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\\ &\subseteq \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) \triangle \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\end{aligned} \tag{96}$$

and since $\mathscr{P}^{(j-1)}$ respects $\mathcal{H}^{(j-1)}$ (cf. part (*b1*)) and $\mathscr{P}^{(j-1)}$ respects $\tilde{\mathcal{G}}^{(j-1)}$ (cf. Setup 36),

$$\begin{aligned}&\bigcup\Big\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) \triangle \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\Big\}\\ &\qquad = \mathcal{K}_j^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_j^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\,.\end{aligned} \tag{97}$$

Combining (96) and (97) with the induction hypothesis on (*b2*) for $j-1$ yields

$$\begin{aligned}&\Big|\bigcup\Big\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}_-^{(j-1)}[\Lambda_j] \setminus \hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\}\Big|\\ &\qquad\qquad \leq \Big|\mathcal{K}_j^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_j^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\Big| < \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\,.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\bigg|\bigcup_{s\in[t]}\Big\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big\}\bigg| &= \Big|\bigcup\Big\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\}\Big|\\ &\geq \sum\Big\{\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\Big| \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}_-^{(j-1)}[\Lambda_j]\Big\} - \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\,.\end{aligned}$$

Applying (71) of Fact 43 to each term in the sum above yields the further lower bound

$$\Big|\hat{B}_-^{(j-1)}[\Lambda_j]\Big|\,(1-\tilde{\eta})\left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j - \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\,.$$

Finally, from our assumption (92) and inequality (31), we infer

$$\begin{aligned}\bigg|\bigcup_{s\in[t]}\Big\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big\}\bigg| &\geq \binom{\ell}{j}^{-1}(1-\tilde{\eta})\sqrt{\delta_j}\left(\prod_{h=2}^{j-1}\Big(d_h b_h \tilde{d}_h\Big)^{\binom{j}{h}}\right) n^j - \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\\ &\geq \left(\binom{\ell}{j}^{-1}(1-\tilde{\eta})(1-\nu)\sqrt{\delta_j} - \delta_{j-1}^{1/3}\right)\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j \geq \delta_j^{3/4}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\end{aligned} \tag{98}$$

where the last inequality follows from the choice of $\tilde{\eta}$, ν, and $\delta_j \ggg \delta_{j-1}$. Now, (94) follows from (98) combined with (68) of Fact 42 for $j-1$ and $i = j$.

It is left to verify (95). First, observe that from the definition of $\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}$ and (79), we have

$$\begin{aligned}\Big|\tilde{\mathcal{G}}^{(j)} \cap \bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| &= \sum\Big\{\Big|\tilde{\mathcal{G}}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\Big| \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\} \\ &= \sum\sum\Big\{\Big|\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)\Big| \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j],\ \beta \in J(\hat{\boldsymbol{x}}^{(j-1)})\Big\}.\end{aligned} \tag{99}$$

Recall that by Definition 29, part (ii), every $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)$ is $(\tilde{\delta}_j, \tilde{d}_j, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j]$ and $\beta \in J(\hat{\boldsymbol{x}}^{(j-1)})$. Consequently, from (71) of Fact 43, we note

$$\Big|\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)\Big| \le \Big(\tilde{d}_j + \tilde{\delta}_j\Big)(1+\tilde{\eta})\left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j$$

for every $\hat{\boldsymbol{x}}^{(j-1)}$ and β considered in (99). Consequently, we may bound (99) using that for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j] \subseteq \hat{B}_-^{(j-1)}$, $|J(\hat{\boldsymbol{x}}^{(j-1)})| < (1-\sqrt{\delta_j})d_j b_j$ (cf. (91)) as

$$\Big|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big| \times \Big(1-\sqrt{\delta_j}\Big) d_j b_j \times \Big(\tilde{d}_j + \tilde{\delta}_j\Big)(1+\tilde{\eta})\left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j . \tag{100}$$

On the other hand, we infer again from (71) of Fact 43 that

$$\begin{aligned}\Big|\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| &= \sum\Big\{\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\Big| \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big\} \\ &\ge \Big|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\Big| \times (1-\tilde{\eta})\left(\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j .\end{aligned} \tag{101}$$

Comparing (100) and (101) yields

$$d\left(\tilde{\mathcal{G}}^{(j)} \big| \hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j \frac{\big(1-\sqrt{\delta_j}\big)\big(b_j\tilde{d}_j + b_j\tilde{\delta}_j\big)(1+\tilde{\eta})}{1-\tilde{\eta}} .$$

From (31) and $\tilde{\eta} \lll \delta_j$ (observe $j > 2$ here), we infer

$$d\left(\tilde{\mathcal{G}}^{(j)} \big| \hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j\left(1-\delta_j^{3/4}\right)\left(1+\nu+b_j\tilde{\delta}_j\right) . \tag{102}$$

Finally, we observe that by Definition 29 (i) and $\tilde{d}_j > \tilde{\delta}_j$ we have $b_j < 2/\tilde{d}_j$. Therefore, (95) follows from (102) and the choice of constants $\delta_j \ggg \nu \ggg \tilde{d}_j \ggg \tilde{\delta}_j$. This completes the proof of Claim 45′. □

6.2.3. *Proof of Property (c) of Lemma 37.* In this section, we define the promised hypergraph $\mathcal{H}^{(k)}$ and confirm the Properties $(c1)$ and $(c2)$. We first observe that the hypergraph $\tilde{\mathcal{G}}^{(k)}$ 'almost' satisfies the properties of the promised $\mathcal{H}^{(k)}$. In particular, due to Lemma 30 (i), the hypergraph $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular w.r.t. every polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$. However, the relative density $d\big(\tilde{\mathcal{G}}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ of $\tilde{\mathcal{G}}^{(k)}$ w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ may be 'wrong' (that is, differing substantially from d_k) for some $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$. We intend to replace $\tilde{\mathcal{G}}^{(k)}$ on those polyads. To that end, define

$$\hat{B}^{(k-1)} = \Big\{\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\big(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\big) \colon \Big|d\left(\tilde{\mathcal{G}}^{(k)}\Big|\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\right) - d_k\Big| > \sqrt{\delta_k}\Big\}.$$

Similarly as in Section 6.2.1 (cf. Claim 45), we claim $\Big|\hat{B}^{(k-1)}\Big|$ is small.

Claim 46. $\Big|\hat{B}^{(k-1)}\Big| < 2\sqrt{\delta_k}\prod_{h=2}^{k-1}(d_h b_h)^{\binom{k}{h}} \times b_1^k$.

The proof of Claim 46 follows the lines of the proof of Claim 45. Observe that in the proof of Claim 45, we used the Counting Lemma for $(j-1)$-uniform hypergraphs (cf. (71) of Fact 43). In a proof of Claim 46, we would do the same with $j-1 = k-1$.

We prepare to define $\mathcal{H}^{(k)}$. To that end, we first define auxiliary hypergraphs $\mathcal{S}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ for $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. While our work below is straightforward, we do need to distinguish two cases depending on whether $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$ or $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(k-1)}$.

Case 1 ($\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \setminus \hat{B}^{(k-1)}$)**.** We set

$$\mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = \tilde{\mathcal{G}}^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\,. \tag{103}$$

Case 2 ($\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$)**.** Observe that

$$\big|\mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\big| > \frac{(n/b_1)^k}{\ln(n/b_1)}$$

by (71) of Fact 43. Therefore, we may apply the Slicing Lemma, Lemma 31, with $m = n/b_1$, $p = d_k$, $\varrho = 1$, $\delta = \tilde{\delta}_k/3$ and $r_{\mathrm{SL}} = \tilde{r}$, and conclude that for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$ there exists a hypergraph

$$\mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) \subseteq \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) \tag{104}$$

which is $(\tilde{\delta}_k, d_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

We now define the promised hypergraph $\mathcal{H}^{(k)}$ as

$$\mathcal{H}^{(k)} = \bigcup\Big\{\mathcal{S}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})\colon\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})\Big\}\,. \tag{105}$$

With $\mathcal{H}^{(k)}$ defined above, we claim that property (*c1*) of Lemma 37 is immediately satisfied. Indeed, Property (*c1*) is clearly satisfied whenever $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$. On the other hand, by Property (i) of Lemma 30, for any $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\left(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\right)$, $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. Moreover, by the definition of $\hat{B}^{(k-1)}$ above and $\mathcal{H}^{(k)}$ in (105), for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \setminus \hat{B}^{(k-1)}$,

$$d\left(\mathcal{H}^{(k)}\middle|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\right) = d\left(\mathcal{S}^{(k)}\middle|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\right) = d\left(\tilde{\mathcal{G}}^{(k)}\middle|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\right) = d_k \pm \sqrt{\delta_k}\,.$$

Thus, property (*c1*) is satisfied with $\mathcal{H}^{(k)}$ as defined above. The remainder of this section is therefore devoted to the proof of property (*c2*) for $\mathcal{H}^{(k)}$.

The proof of property (*c2*) is similar (but somewhat simpler) than the proof of (*b2*). Here, we partition the ℓ-tuples $L_0 \in \mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})$ into

$$\begin{aligned}
\mathfrak{K}_\ell^{(k)}(1) &= \left\{L_0 \in \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right)\colon\ \exists\, K_0 \in \binom{L_0}{k} \text{ s.t. } K_0 \in \left(\mathcal{K}_k^{(k-1)}(\mathcal{H}^{(k-1)}) \triangle \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})\right)\right\}\\
\mathfrak{K}_\ell^{(k)}(2) &= \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right)\Big\backslash\, \mathfrak{K}_\ell^{(k)}(1) =\\
&= \left\{L_0 \in \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right)\colon\ \forall\, K_0 \in \binom{L_0}{k}\quad K_0 \in \left(\mathcal{K}_k^{(k-1)}(\mathcal{H}^{(k-1)}) \cap \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})\right)\right\}.
\end{aligned}$$

The last equality follows from an argument similar to the one given after (82).

Let L_0 be in $\mathfrak{K}_\ell^{(k)}(2)$. Observe that $L_0 \in \mathcal{K}_\ell^{(k-1)}(\mathcal{H}^{(k-1)}) \cap \mathcal{K}_\ell^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})$ (i.e., every $K_0 \in \binom{L_0}{k}$ 'touches' polyads $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$ only) and $L_0 \in \mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})$. Recall $\mathcal{H}^{(k)}$ and $\tilde{\mathcal{G}}^{(k)}$ only differ on 'bad' polyads (see the construction of $\mathcal{H}^{(k)}$ in (104)–(105)). Consequently, there exists $K_0 \in \binom{L_0}{k}$ that 'touches' some 'bad' polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$. Summarizing the above, we observe

$$\begin{aligned}
\mathfrak{K}_\ell^{(k)}(2) = \Bigg\{L_0 \in \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right)\colon\ \exists\, K_0 \in \binom{L_0}{k}\quad \text{and}&\\
\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)} \text{ so that } K_0 \in \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\Bigg\}&.
\end{aligned} \tag{106}$$

By definition, $\mathfrak{K}_\ell^{(k)}(1) \cup \mathfrak{K}_\ell^{(k)}(2)$ is a partition of $\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})$ and so we have

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big)\right| = \big|\mathfrak{K}_\ell^{(k)}(1)\big| + \big|\mathfrak{K}_\ell^{(k)}(2)\big|\,. \tag{107}$$

We now bound $\big|\mathfrak{K}_\ell^{(k)}(1)\big|$ and $\big|\mathfrak{K}_\ell^{(k)}(2)\big|$ to obtain part (c) of Lemma 37.

Bounding $|\mathfrak{K}_\ell^{(k)}(1)|$. The upper bound again easily follows from $(b2)$ of Lemma 37 for $j = k-1$ and $i = \ell$. Indeed, observe $\mathfrak{K}_\ell^{(k)}(1) \subseteq \mathcal{K}_\ell^{(k-1)}\big(\mathcal{H}^{(k-1)}\big) \triangle \mathcal{K}_\ell^{(k-1)}\big(\tilde{\mathcal{G}}^{(k-1)}\big)$ holds by the same argument presented after (85). We therefore see

$$\big|\mathfrak{K}_\ell^{(k)}(1)\big| \le \delta_{k-1}^{1/3} \left(\prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}}\right) n^\ell \le \frac{1}{2}\delta_k^{1/3} \left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell\,, \tag{108}$$

where the last inequality follows from $\delta_{k-1}^{1/3} \lll \delta_k^{1/3} d_k$ as given in Figure 2.

Bounding $|\mathfrak{K}_\ell^{(k)}(2)|$. As a consequence of $(b1)$ for $2 \le j < k$ and $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$, we infer there are $\prod_{h=2}^{k-1}(d_h b_h)^{\binom{\ell}{h}-\binom{k}{h}} \times b_1^{\ell-k}$ ways to complete $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ to a $\big((\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\dots,\tilde{d}_{k-1}),\tilde{r}\big)$-regular $(n/b_1,\ell,k-1)$-subcomplex $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{k-1})} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{k-1}$ of $\mathcal{H}^{(\boldsymbol{k-1})}$. Note that (70) of Fact 43 applied with $i = \ell$ and $j = k$ yields

$$\Big|\mathcal{K}_\ell^{(k-1)}(\check{\mathcal{H}}^{(k-1)})\Big| \le (1+\tilde{\eta}) \left(\prod_{h=2}^{k-1} \tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell \le 2\left(\prod_{h=2}^{k-1} \tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell$$

for each such $\check{\mathcal{H}}^{(k-1)}$. Since this holds for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$, the last inequality combined with (106) and Claim 46 yields

$$\begin{aligned}\Big|\mathfrak{K}_\ell^{(k)}(2)\Big| &\le \Big|\hat{B}^{(k-1)}\Big| \times \left(\prod_{h=2}^{k-1}(d_h b_h)^{\binom{\ell}{h}-\binom{k}{h}}\right) b_1^{\ell-k} \times 2\left(\prod_{h=2}^{k-1} \tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell \\ &\le 4\sqrt{\delta_k}\left(\prod_{h=2}^{k-1}(d_h b_h \tilde{d}_h)^{\binom{\ell}{h}}\right) n^\ell \overset{(31)}{\le} 4\sqrt{\delta_k}(1+\nu)\left(\prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}}\right) n^\ell\,,\end{aligned} \tag{109}$$

and so by the choice of $\delta_k \lll d_k$ we have

$$\Big|\mathfrak{K}_\ell^{(k)}(2)\Big| \le \frac{1}{2}\delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell\,. \tag{110}$$

Combining (107) with (108) and (110) yields part $(c2)$ of Lemma 37.

6.3. **Proof of Lemma 38.** Lemma 38 follows from a simple and straightforward application of the Slicing Lemma, Lemma 31. Recall Setup 36 and that $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(h)}\}_{h=1}^k$ is the (n,ℓ,k)-complex given by Lemma 37.

Proof of Lemma 38. Recall that by part (c) of Lemma 37, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, the (n,ℓ,k)-cylinder

$$\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) \text{ is } (\tilde{\delta}_k, \bar{d}(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r})\text{-regular} \tag{111}$$

w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ where $\bar{d}(\hat{\boldsymbol{x}}^{(k-1)}) = d_k \pm \sqrt{\delta_k}$.

Construction of $\boldsymbol{\mathcal{H}}_{\boldsymbol{-}}$. For $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, apply the Slicing Lemma, Lemma 31, with $\varrho = \bar{d}(\hat{\boldsymbol{x}}^{(k-1)})$, $p = (d_k - \sqrt{\delta_k})/\varrho$, $\delta = \tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to $\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ to obtain a $(3\tilde{\delta}_k, d_k - \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\mathcal{S}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$.

Note that the assumptions of the Slicing Lemma are satisfied. This is due to the fact that the family of partitions $\mathscr{P}$ is an almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family and, consequently, the polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $\big((\tilde{\delta}_2,\dots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\dots,\tilde{d}_{k-1}),\tilde{r}\big)$-regular. Hence, by (71) of Fact 43 (with $j = k$),

$$\Big|\mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\Big| > \frac{(n/b_1)^k}{\ln(n/b_1)}$$

for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$.

We then set

$$\mathcal{H}_-^{(k)} = \bigcup\Big\{\mathcal{S}_-^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) \colon \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})\Big\}\,.$$

Obviously, $\mathcal{H}_-^{(k)}$ has the desired properties (α) and $(\beta 1)$ by construction.

Construction of $\mathcal{H}_+$. The construction of $\mathcal{H}_+^{(k)}$ is similar and follows by an application of the Slicing Lemma to the complement of $\mathcal{H}^{(k)}$. More precisely, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, set $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{K}_k^{(k-1)}(\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(k-1)}))\setminus\mathcal{H}^{(k)}$. Note that, due to (111), $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(\tilde{\delta}_k, 1-\bar{d}(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r})$-regular. Consequently, we can apply the Slicing Lemma to $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\varrho = 1 - \bar{d}(\hat{\boldsymbol{x}}^{(k-1)})$, $p = (1 - d_k - \sqrt{\delta_k})/\varrho$, $\delta = \tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to obtain a $(3\tilde{\delta}_k, 1 - d_k - \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\overline{\mathcal{S}}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. We then set $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{K}_k^{(k-1)}(\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(k-1)})) \setminus \overline{\mathcal{S}}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Clearly, $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(3\tilde{\delta}_k, d_k + \sqrt{\delta_k}, \tilde{r})$-regular and $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \supseteq \mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Finally, we define $\mathcal{H}_+^{(k)}$ to be the union of all $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ constructed that way.

Construction of $\mathcal{F}$. The construction of $\mathcal{F}^{(k)}$ is more involved owing to the requirement $\mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}$.

Let $\mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)}$ be given as constructed above and for $* \in \{+,-\}$ and $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, let $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}_*^{(k)} \cap \mathcal{K}_k^{(k-1)}(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}))$. Due to ($\beta$1) and ($\beta$2), $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(3\tilde{\delta}_k, d_k^*, \tilde{r})$-regular where $d_k^- = d_k - \sqrt{\delta_k}$ and $d_k^+ = d_k + \sqrt{\delta_k}$. Moreover, $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \supseteq \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ and, consequently, $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \setminus \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(6\tilde{\delta}_k, 2\sqrt{\delta_k}, \tilde{r})$-regular. We now apply the Slicing lemma to $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \setminus \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\varrho = 2\sqrt{\delta_k}$, $p = \sqrt{\delta_k}/\varrho = 1/2$, $\delta = 6\tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to obtain a $(18\tilde{\delta}_k, \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\mathcal{S}_{\mathcal{F}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Now define $\mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ to be the disjoint union $\mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \cup \mathcal{S}_{\mathcal{F}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Clearly, $\mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Moreover, it is straightforward to verify that $\mathcal{F}^{(k)}$ is $(21\tilde{\delta}_k, d_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ and, consequently,

$$\mathcal{F}^{(k)} = \bigcup \left\{ \mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \colon \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \right\},$$

has the desired properties. □

7. Proof of the Union Lemma

7.1. **Union of regular hypergraphs.** Below we present some useful facts regarding regularity properties of the union of regular (m,j,j)-cylinders. We distinguish two cases depending whether the (m,j,j)-cylinders in question have the same underlying polyad or not.

The first proposition says that we may unite disjoint regular (m,j,j)-cylinders of the same density which share the same underlying $(m,j,j-1)$-cylinder without spoiling the regularity too much.

Proposition 47. *Let $j \geq 2$, t and m be fixed positive integers, let δ and d be positive reals and let $\mathcal{P}_1^{(j)}, \ldots, \mathcal{P}_t^{(j)}$ be a family of pairwise edge disjoint (m,j,j)-cylinders with the same underlying $(m,j,j-1)$-cylinder $\hat{\mathcal{P}}^{(j-1)}$. If for every $s \in [t]$, the hypergraph $\mathcal{P}_s^{(j)}$ is $(\delta, d, 1)$-regular with respect to $\hat{\mathcal{P}}^{(j-1)}$, then $\mathcal{P}^{(j)} = \bigcup_{s\in[t]} \mathcal{P}_s^{(j)}$ is $(t\delta, td, 1)$-regular with respect to $\hat{\mathcal{P}}^{(j-1)}$.*

The proof of Proposition 47 is straightfoward and short and we therefore omit it. The next proposition gives us control when we unite hypergraphs having different underlying polyads. Before we make this precise, we define the setup for our proposition.

Setup 48. *Let $j \geq 3$, t and m be fixed positive integers and let δ and d be positive reals. Let $\{\hat{\mathcal{P}}_s^{(j-1)}\}_{s\in[t]}$ be a family of $(m,j,j-1)$-cylinders such that*

$$\begin{gathered}\mathcal{K}_j^{(j-1)}\left(\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}\right) = \bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}_s^{(j-1)}) \quad \textit{and} \\ \mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}_s^{(j-1)}) \cap \mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}_{s'}^{(j-1)}) = \varnothing \quad \textit{for } 1 \leq s < s' \leq t\,.\end{gathered} \tag{112}$$

From (112), $\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}_s^{(j-1)})$ is a partition of the j-cliques of $\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$. Let $\{\mathcal{P}_s^{(j)}\}_{s\in[t]}$ be a family of (m,j,j)-cylinders such that $\hat{\mathcal{P}}_s^{(j-1)}$ underlies $\mathcal{P}_s^{(j)}$ for any $s \in [t]$. Set $\hat{\mathcal{P}}^{(j-1)} = \bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$ and $\mathcal{P}^{(j)} = \bigcup_{s\in[t]} \mathcal{P}_s^{(j)}$.

Proposition 49. *Let* $\{\mathcal{P}_s^{(j)}\}_{s\in[t]}$ *and* $\{\hat{\mathcal{P}}_s^{(j-1)}\}_{s\in[t]}$ *satisfy Setup 48. If* $\mathcal{P}_s^{(j)}$ *is* $(\delta, d, 1)$*-regular w.r.t.* $\hat{\mathcal{P}}_s^{(j-1)}$ *for every* $s \in [t]$*, then* $\mathcal{P}^{(j)}$ *is* $(2\sqrt{\delta}, d, 1)$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}$*.*

Proof. Let $\hat{\mathcal{Q}}^{(j-1)} \subseteq \hat{\mathcal{P}}^{(j-1)}$ be such that

$$\big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\big| \geq \sqrt{\delta}\big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\big|. \tag{113}$$

For every $s \in [t]$, set $\hat{\mathcal{Q}}_s^{(j-1)} = \hat{\mathcal{Q}}^{(j-1)} \cap \hat{\mathcal{P}}_s^{(j-1)}$. Since $\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big)$ is a partition of the j-cliques of $\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$, $\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)$ is a partition of the j-cliques of $\hat{\mathcal{Q}}^{(j-1)} = \bigcup_{s\in[t]} \hat{\mathcal{Q}}_s^{(j-1)}$. As such,

$$\sum_{s\in[t]} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| = \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big|. \tag{114}$$

Define

$$T = \Big\{s \in [t]\colon \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \geq \delta\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big)\Big|\Big\}.$$

Observe that

$$\sum_{s\notin T} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| < \delta \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\Big| \overset{(113)}{\leq} \sqrt{\delta}\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big|. \tag{115}$$

Consequently (113), (114) and (115) give

$$\sum_{s\in T} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \geq \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| - \delta\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\Big| \geq \big(1-\sqrt{\delta}\big)\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big|. \tag{116}$$

If $s \in T$, then the $(\delta, d, 1)$-regularity of $\mathcal{P}_s^{(j)}$ w.r.t. $\hat{\mathcal{P}}_s^{(j-1)}$ implies

$$\Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| = (d \pm \delta)\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big|.$$

Consequently,

$$\begin{aligned}\Big|\mathcal{P}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| &= \sum_{s\in[t]} \Big|\mathcal{P}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \\ &= \sum_{s\in T} \Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| + \sum_{s\notin T} \Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \\ &= (d\pm\delta)\sum_{s\in T} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| + \sum_{s\notin T} \Big|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big|.\end{aligned}$$

We then see

$$(d-\delta)\sum_{s\in T} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \leq \Big|\mathcal{P}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| \leq (d+\delta)\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big| + \sum_{s\notin T} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big|.$$

In view of (115) and (116), we infer

$$(d-\delta)(1-\sqrt{\delta}) \leq d\big(\mathcal{P}^{(j)}\big|\hat{\mathcal{Q}}^{(j-1)}\big) \leq d + \delta + \sqrt{\delta}$$

from which Proposition 49 follows. □

7.2. **Proof of Lemma 40.** Before proving Lemma 40, we recall some notation. For $* \in \{+, -\}$, let $\boldsymbol{\mathcal{H}_*} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \cup \mathcal{H}_*^{(k)} = \{\mathcal{H}_*^{(j)}\}_{j=1}^{k}$ be given by Lemma 38. It follows that for each $j = 2, \ldots, k-1$, the set $\hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$ of polyad addresses with $\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq \mathcal{H}^{(j-1)}$ satisfies that for each $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$, there is an index set $I(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq [b_j]$ of size $d_j b_j$ such that

$$\mathcal{H}_*^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha\in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big).$$

Moreover, $d\big(\mathcal{H}_*^{(k)}\big|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) = d_k^*$, where d_k^* is defined in (36). Recall that for $\Lambda_j = (\lambda_1, \ldots, \lambda_j) \in \binom{[\ell]}{j}_<$, we denote by $\mathcal{H}_*^{(j)}[\Lambda_j]$ the subhypergraph of $\mathcal{H}_*^{(j)}$ induced on $V_{\lambda_1} \cup \cdots \cup V_{\lambda_j}$.

Due to Lemma 37, we know that for $j = 2, \ldots, k-1$, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, the set $I(\hat{\boldsymbol{x}}^{(j-1)})$ satisfies $|I(\hat{\boldsymbol{x}}^{(j-1)})| = d_j b_j$; moreover, for every $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})$, the $(n/b_1, j, j)$-cylinder $\mathcal{P}^{(j)}\big(\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ is $(\tilde{\delta}, \tilde{d}, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Inductively on j, we aim to show that $\mathcal{H}_*^{(j)}[\Lambda_j]$, which is the union of all $\mathcal{P}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}, \alpha)$, with $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{j-1})$, $\hat{\boldsymbol{x}}_0 = \Lambda_j$ and $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})$, is regular w.r.t. $\mathcal{H}_*^{(j-1)}[\Lambda_j]$.

Proof of Lemma 40. We only prove the statement about $\boldsymbol{\mathcal{H}}_*$ here. The proof for $\boldsymbol{\mathcal{F}}$ is identical. Consider the following statement:

(S_j) If $\Lambda_j = (\lambda_1, \ldots, \lambda_j) \in \binom{[\ell]}{j}_<$ then $\boldsymbol{\mathcal{H}}_*^{(j)}[\Lambda_j]$ is a $\big((\varepsilon', \ldots, \varepsilon'), (d_2^*, \ldots, d_j^*), 1\big)$-regular (n, j, j)-complex.

Lemma 40 then follows from (S_k).

We prove statement (S_j) by induction on j. Suppose $j = 2$ and let $\Lambda_2 \in \binom{[\ell]}{2}_<$ be given. By (a) of Lemma 37, we have $\tilde{\mathcal{G}}^{(2)} = \mathcal{H}^{(2)} = \mathcal{H}_*^{(2)}$. Consequently, $\mathcal{H}_*^{(2)}[\Lambda_2]$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{H}_*^{(1)}[\Lambda_2]$ and (S_2) follows from $\delta_2 \lll \varepsilon'$ (cf. Figure 2).

We now proceed to the induction step. Assume $3 \le j \le k$ and (S_{j-1}) holds. Let $\Lambda_j = (\lambda_1, \ldots, \lambda_j) \in \binom{[\ell]}{j}_<$ be arbitrary but fixed. The proof of (S_j) consists of three steps and we begin with the easiest.

Step 1. Let $\hat{X}(\Lambda_j) \subseteq \hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$ be the set of all polyad addresses $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{j-1})$ with $\hat{\boldsymbol{x}}_0 = \Lambda_j$. In Step 2, we consider

$$\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \tag{117}$$

for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$. To that end, fix a $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$. This (and only this) step splits into two cases, depending on whether $j = k$ or not.

Case 1 $(3 \le j < k)$**.** We apply Proposition 47 to

$$\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \qquad \text{and} \qquad \Big\{\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) \colon \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})\Big\}$$

with $\delta = \delta_j$, $d = \tilde{d}_j$ (since $j < k$), and $t = \big|I(\hat{\boldsymbol{x}}^{(j-1)})\big| = d_j b_j = d_j^* b_j$. Consequently, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$,

$$\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) = \mathcal{H}_*^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$$

is $\big((d_j^* b_j)\tilde{\delta}_j, (d_j^* b_j)\tilde{d}_j, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Since $b_j \tilde{\delta}_j \le 2\tilde{\delta}_j / \tilde{d}_j \lll \nu$ and from (31), each such $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $\big(2\nu, d_j^*, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

Case 2 $(j = k)$**.** Here, we infer from (β) of Lemma 38 that $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}_*^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ is $(3\tilde{\delta}_k, d_k^*, 1)$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. Hence, $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is also $\big(2\nu, d_k^*, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

Summarizing the two cases, we infer that for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$, the $(n/b_1, j, j)$-cylinder $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$, as given in (117), is

$$\big(2\nu, d_j^*, 1\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\,. \tag{118}$$

Step 2. In this step, we apply the induction assumption (S_{j-1}). For every $\iota \in [j]$, set

$$\Lambda_j\,(\iota) = (\lambda_1, \ldots, \lambda_{\iota-1}, \lambda_{\iota+1}, \ldots, \lambda_j)\,.$$

We apply (S_{j-1}) to the $(n, j, j-1)$-complex $\boldsymbol{\mathcal{H}}_*^{(j-1)}[\Lambda_j(\iota)]$ for every $\iota \in [j]$. As a result, we infer that

$$\boldsymbol{\mathcal{H}}_*^{(j-1)}[\Lambda_j] = \bigcup_{\iota \in [j]} \boldsymbol{\mathcal{H}}_*^{(j-1)}[\Lambda_j\,(\iota)] = \Big\{\bigcup_{\iota \in [j]} \mathcal{H}^{(h)}[\Lambda_j\,(\iota)]\Big\}_{h=1}^{j-1}$$

is an $\big((\varepsilon', \ldots, \varepsilon'), (d_2^*, \ldots, d_{j-1}^*), 1\big)$-regular $(n, j, j-1)$-complex.

Step 3. Finally, as the last step, we show that the disjoint union

$$\bigcup_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)} \mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j)}[\Lambda_j]$$

is $(\varepsilon', d_j^*, 1)$-regular w.r.t. $\mathcal{H}_*^{(j-1)}[\Lambda_j]$ (see (117)). Recall that $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $(2\nu, d_j^*, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ for each $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$, as shown in Step 1 (cf. (118)). A moment of thought yields that $\{\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\}_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)}$ satisfies (112). Consequently, the assumptions of Proposition 49 are satisfied with $\delta = 2\nu$, $d = d_j^*$, and $t = |\hat{X}(\Lambda_j)|$ for the families

$$\left\{\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)} \quad \text{and} \quad \left\{\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)}.$$

Therefore, it follows from Proposition 49 that $\mathcal{H}_*^{(j)}[\Lambda_j] = \bigcup_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)} \mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $(2\sqrt{2\nu}, d_j^*, 1)$-regular w.r.t. $\bigcup_{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)} \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j-1)}[\Lambda_j]$. Then (S_j) follows from $2\sqrt{2\nu} < \varepsilon'$ (cf. Figure 2). □

8. Concluding Remarks

In the course of writing this paper, the authors realized that the methods used here, combined with the result of [40], yields a hypergraph regularity lemma that would be simpler to state and likely more convenient to use. For 3-uniform hypergraphs, such a result immediately follows from Theorem 34 and the RS-Lemma with $k = 3$ (equivalently, the FR-Lemma).

Theorem 50. *For every positive real ν and a positive real-valued function $\varepsilon(D_2)$, there exist integers L_3 and n_3 such that for any 3-uniform hypergraph $\mathcal{H}^{(3)}$ on $n \geq n_3$ vertices there exists a 3-uniform hypergraph $\mathcal{F}^{(3)}$ and a $(\nu, \varepsilon(d_2), d_2, 1)$-equitable family of partitions $\mathscr{R} = \mathscr{R}(2, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$ such that*

(i) *for every $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$*

$$\left|(\mathcal{H}^{(3)} \triangle \mathcal{F}^{(3)}) \cap \mathcal{K}_3^{(2)}(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)}))\right| \leq \nu d_2^3 \left(\frac{n}{a_1}\right)^3, \tag{119}$$

(ii) *all but at most νn^3 edges of $K_n^{(3)}$ belong to some polyad $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$ where $\mathcal{F}^{(3)}$ is $(\varepsilon(d_2), 1)$-regular w.r.t. $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$, and*

(iii) $\operatorname{rank} \mathscr{R} \leq L_3$.

We omit the techincal details of the proof of Theorem 50 (a proof based Theorem 34 and the RS-Lemma with $k = 3$ is given in [42, Chapter 10]).

There is an important single difference between Theorem 50 and the FR-Lemma (or, equivalently, RS-Lemma with $k = 3$); Theorem 50 provides an environment sufficient for a direct application of the Dense Counting Lemma, Theorem 16. Indeed, unlike the the output of the FR-Lemma where one has constants δ_3, d_2, $\delta_2(d_2)$ and r, so that $\delta_3 \gg d_2 \gg \delta_2(d_2), 1/r$, Theorem 50 admits sufficiently small $\varepsilon(d_2)$ with $\varepsilon(d_2) \ll d_2$ and no formulation of r such that both the 3-uniform hypergraph $\mathcal{F}^{(3)}$ and the graphs from the underlying partition $\mathscr{R}^{(2)}$ are $\varepsilon(d_2)$-regular.

The only cost of the cleaner environment Theorem 50 renders is that our original input hypergraph $\mathcal{H}^{(3)}$ is slightly (albeit negligibly) altered to the output hypergraph $\mathcal{F}^{(3)}$.

In [36] we prove a generalization of Theorem 50 to k-uniform hypergraphs. The proof of that generalization is more technical and requires a restructuring of the arguments from [40] and this paper.

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