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The size-Ramsey number of trees

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THE SIZE-RAMSEY NUMBER OF TREES

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Abstract. Given a graph G, the size-Ramsey number $\hat{r}(G)$ is the minimum number m for which there exists a graph F on m edges such that any two-coloring of the edges of F admits a monochromatic copy of G.

In 1983, J. Beck introduced an invariant $\beta(\cdot)$ for trees and showed that $\hat{r}(T) = \Omega(\beta(T))$. Moreover he conjectured that $\hat{r}(T) = \Theta(\beta(T))$. We settle this conjecture by providing a family of graphs and an embedding scheme for trees.

1. Introduction

For graphs G and H, the size-Ramsey number $\hat{r}(G,H)$, introduced by Erdős et al. [3], is the smallest number m such that there exists a graph F on m edges with the property that, in any red-blue coloring of the edges of F, there exists either a red copy of G or a blue copy of H.

For a real number $\alpha \in [0,1]$ and graphs F, G we shall write $F \to_\alpha G$ if any subgraph $F' \subseteq F$ with $e(F') \geq \alpha e(F)$ contains a copy of G as a subgraph. Notice that if $F \to_{1/2} G$ then $\hat{r}(G) = \hat{r}(G,G) \leq e(F)$.

It is well known that $\hat{r}(K_n)$ grows exponentially with n. In contrast, Beck [1], answering a question of Erdős, showed that for P_t, the path on t vertices, we have

$$\hat{r}(P_t) = \hat{r}(P_t, P_t) \leq 900t.$$

In fact, Beck proved that for any $\alpha \in (0,1]$ there is $c = c(\alpha)$ such that **a.a.s.** (asymptotically almost surely) the random graph $G = G_{n,c/n}$ satisfies $G \to_\alpha P_{\lfloor n/c \rfloor}$.

Friedman and Pippenger [5] improved this result by showing that any tree with maximum degree Δ and t vertices has size-Ramsey number $c(\Delta)t$, where $c(\Delta) = O(\Delta^4)$. This was later improved to $c(\Delta) = O(\Delta^2)$ by Ke [8] and to $c(\Delta) = O(\Delta)$ by Haxell and Kohayakawa [6].

Although certain trees $\mathcal{T}$ have size-Ramsey of order $\Delta(\mathcal{T})\,|\mathcal{T}|$, it is clear that the size-Ramsey of the star $K_{1,t}$ is not of order t^2. Indeed, $K_{1,\alpha^{-1}t} \to_\alpha K_{1,t}$ for any $\alpha \in (0,1]$. Hence, the bound $\Delta(\mathcal{T})\,|\mathcal{T}|$ may be far from optimal in many cases.

In [2], Beck introduced the tree invariant $\beta(\mathcal{T})$ which is defined as follows. Let $V(\mathcal{T}) = V_0(\mathcal{T}) \cup V_1(\mathcal{T})$ be the partition determined by the unique proper two-coloring of the vertex set of $\mathcal{T}$. Set $\Delta_i = \Delta_i(\mathcal{T}) = \max\{d_{\mathcal{T}}(v) : v \in V_i(\mathcal{T})\}$ and $n_i = n_i(\mathcal{T}) = |V_i(\mathcal{T})|$ for $i = 0,1$ and let $\beta(\mathcal{T}) = n_0\Delta_0 + n_1\Delta_1$. Improving his previous result, Beck [2] proved that for any tree $\mathcal{T}$,

$$\beta(\mathcal{T})/4 \leq \hat{r}(\mathcal{T}) \leq O\big(\beta(\mathcal{T})(\log|\mathcal{T}|)^{12}\big)$$

and conjectured that $\hat{r}(\mathcal{T}) = O(\beta(\mathcal{T}))$. Haxell and Kohayakawa [6] significantly improved the upper bound to $\hat{r}(\mathcal{T}) = O\big(\beta(\mathcal{T})\log\Delta(\mathcal{T})\big)$.

We settle this conjecture by showing that for any $(n_0, \Delta_0, n_1, \Delta_1)$ and $\alpha \in (0,1]$ there exists N_0, N_1 and $p \in [0,1]$ with $pN_0N_1 = O_\alpha(n_0\Delta_0 + n_1\Delta_1)$ such that **a.a.s.** the random bipartite graph $G = G_{N_0,N_1;p}$ satisfies $G \to_\alpha \mathcal{T}$ for any tree $\mathcal{T}$ with $\Delta_i(\mathcal{T}) \leq \Delta_i$ and $n_i(\mathcal{T}) \leq n_i$, for $i = 0,1$. Since **a.a.s.** G has $O(pN_0N_1)$ edges, the size-Ramsey of any tree $\mathcal{T}$ is of the order of $\beta(\mathcal{T})$.

The embedding of $\mathcal{T}$ into G is done algorithmically. We believe that this algorithmic method is interesting in its own right and that it could be useful in other similar contexts. In fact, in joint work with Kohayakawa, Rödl and Ruciński [7], we have used analogous techniques in an algorithm that embed graphs of bounded degree into sparse random graphs.

1.1. **Organization of the paper.** In order to prove Beck's conjecture we establish several properties that hold **a.a.s.** for random graphs. Any graph satisfying these properties may be used as an upper bound for the size-Ramsey number of trees. However, there is no known graph construction satisfying all these properties. Thus we have resorted to the probabilistic method in order to prove the existence of such graphs. The results on random graphs are stated in Theorem 6 of Section 4.

In Section 6 we exhibit an embedding scheme for trees, Algorithm 1, that finds an isomorphic copy of any tree with prescribed parameters into a graph satisfying the properties listed in Theorem 6.

We shall give an outline of a simpler (somewhat unrealistic) case for the sake of introducing, in an easier context, some of the techniques employed in the general case. This informal outline is given in Section 3.

2. Preliminaries

Given a graph $G = (V, E)$ and disjoint sets $S, T \subset V$, we denote by $E_G(S, T)$ the set of all edges with one endpoint in S and the other endpoint in T and let $e_G(S, T) = |E_G(S, T)|$. The neighborhood of a vertex $v \in V$ is denoted by $\Gamma_G(v)$ and the neighborhood of a set $S \subseteq V$ is denoted by $\Gamma_G(S) = \bigcup_{v \in S} \Gamma_G(v)$.

Definition 1. Given a graph $G = (V, E)$, for any set $S \subseteq V$, we define

$$\Gamma^*_G(S) = \{v \in V \; : \; e_G(\{v\}, S) = 1\}$$

as the set of *unique* neighbors of S. Let $d^*_G(S) = |\Gamma^*(S)|$.

We may omit the subscript if the graph is clear from the context.

If $x, t \in \mathbb{R}$, $\varepsilon > 0$ are such that $x \in [(1-\varepsilon)t, (1+\varepsilon)t]$ then we write $x \sim_\varepsilon t$. We shall also use the standard notations $\Omega_\gamma(f(n))$, $O_\gamma(f(n))$ for the classes of all functions lower/upper bounded by $c(\gamma)f(n)$, where $c = c(\gamma)$ is a constant that only depends on γ. In many computations we implicitly use well-known inequalities such as

$$1 + x \le e^x \quad \text{and} \quad \left(\frac{a}{b}\right)^b \le \binom{a}{b} \le \left(\frac{ea}{b}\right)^b. \tag{1}$$

The Chernoff inequality is also used extensively: for any $\varepsilon > 0$ and any Binomial random variable X with parameters n and p we have

$$\mathbf{P}\big[|X - np| \ge \varepsilon\, np\big] \le \exp\{-\Omega_\varepsilon(np)\}. \tag{2}$$

Definition 2 (LE sets)**.** We say that a set of vertices S in a graph G is *ε-lossless expanding* if $|\Gamma(S) \setminus S| \sim_\varepsilon e(S, V(G) \setminus S)$, that is, almost every edge in the S-cut corresponds to a unique neighbor of S. We may refer to S as an *LE set* for short.

A useful feature of LE sets is their resilience: even if a large fraction of the edges incident to an LE set is removed, the LE property persists. This is stated formally in the following simple lemma.

Lemma 3. *Let G be a graph and $S \subseteq V = V(G)$. For any $G' \subseteq G$ we have*

$$|\Gamma_{G'}(S) \setminus S| \ge e_{G'}(S, V \setminus S) + 2\{|\Gamma_G(S) \setminus S| - e_G(S, V \setminus S)\}.$$

Proof. Let N denote the number of edges $e = uv$ in $E_G(S, V \setminus S)$ such that the end-vertex $v \in V \setminus S$ satisfies $e_G(v, S) \ge 2$. Note that $|\Gamma_G(S) \setminus S| \le \{e_G(S, V \setminus S) - N\} + N/2$, since each edge not counted by N corresponds to exactly one unique neighbor of S and all the edges counted by N may contribute with at most $N/2$ neighbors. We obtain $-N \ge 2\{|\Gamma_G(S) \setminus S| - e_G(S, V \setminus S)\}$. The claim follows as $|\Gamma_{G'}(S) \setminus S| \ge e_{G'}(S, V \setminus S) - N$. ■

Definition 4. Let $\mathcal{T}$ be a tree and $V(\mathcal{T}) = V_0(\mathcal{T}) \cup V_1(\mathcal{T})$ be the canonical bi-partition of $\mathcal{T}$. Set $n_i = |V_i(\mathcal{T})|$ and $\Delta_i = \max\{d_{\mathcal{T}}(v) : v \in V_i(T)\}$, for $i = 0, 1$. The parameter $\beta(\mathcal{T})$ is defined as

$$\beta(\mathcal{T}) = n_0\Delta_0 + n_1\Delta_1.$$

A tree with these parameters is called an $(n_0, \Delta_0, n_1, \Delta_1)$-*tree*.

3. Outline of a simpler case

In this section we consider a simpler, specific case, where we can apply easier versions of the techniques used in the proof of our result. Let us assume that the n_i's and Δ_i's are fixed and satisfy $n_0\Delta_0 = n_1\Delta_1$. Our unrealistic[1] assumption is the existence of a bipartite graph G having classes V_0, V_1 with $100n_i \le |V_i| = N_i = O(n_i)$, $i = 0, 1$, such that all vertices in V_i have degree $D_i = O(\Delta_i)$ and such that for any i and any set $S \subseteq V_i$, with $|S| \le |V_{i-1}|/D_i$ we have $|\Gamma_G(S)| \ge (1-\varepsilon)D_i$ for some small $\varepsilon \ge 0$. In particular, G is a bipartite graph for which we have lossless expansion for essentially all sets (obviously, if S is too large, it cannot expand losslessly).

Next we outline how one could find a copy of an $(n_0, \Delta_0, n_1, \Delta_1)$-tree $\mathcal{T}$ in any sufficiently dense subgraph of G. Suppose that $G' \subseteq G$ is such that $e(G') \ge e(G)/2$. By sequentially removing vertices of low degree, we may ensure that every $v \in V_i' = V_i \cap V(G')$ has degree at least $D_i/4$ and that $e(G') \ge e(G)/4$.

Suppose that f is a partial embedding of $\mathcal{T}$ into G'. A vertex $v \in V' = V(G')$ is *inactive* with respect to f if there is a vertex $u \in V(\mathcal{T})$ such that $v = f(u)$ and, moreover, all neighbors of u are already embedded by f (namely, $f^{-1}(V') \supset \Gamma_{\mathcal{T}}(u)$).

A vertex is called *free* with respect to some partial embedding f if it is neither *reserved* nor in the image of f. We shall describe how a vertex becomes reserved in what follows.

Critical vertices. The main ingredient in the embedding scheme is how to deal with active vertices in G' which have few free neighbors. These vertices will be called *critical.* We associate to every critical vertex v a subset R_v of its free neighborhood which shall be reserved exclusively to embed neighbors of $f^{-1}(v)$ (if v ever gets used in the embedding, otherwise they shall remain unused). In particular, those vertices in R_v will not longer be free.

Let $c \in (0, 1)$ be a fixed constant to be defined later. A vertex from class V_i' $(i = 0, 1)$ is classified as critical if it has less than cD_i free neighbors.

There are basically two difficulties in dealing with critical vertices: since the reserved subsets must be exclusive, they must be disjoint from each other. Moreover, after reserving vertices, one may produce more critical vertices, as those reserved vertices are no longer free. It is therefore essential to make sure that the number of critical vertices is bounded at all times.

To ensure that there are not too many critical vertices, the set of reserved vertices for each critical vertex is relatively small—it has size Δ_0 or Δ_1, depending on which class the critical vertex belongs. Therefore, for each new critical vertex, we reserve a small number of vertices (making then non-free). On the other hand, every critical vertex must send a considerable fraction of its edges into the set of non-free vertices. By the LE property and Lemma 3, the set of critical vertices must be small, otherwise the expansion of the LE set of critical vertices would contradict the fact that the set of non-free vertices is not large.

More formally, let C_i be the set of critical vertices in the class V_i. The number of non-free vertices in V_{1-i} is at most $n_{1-i} + |C_i|\,\Delta_i$. However, every vertex $v \in C_i$ sends at least $d_{G'}(v) - cD_i \ge (1/4-c)D_i$ edges into the set of non-free vertices of V_{1-i}. If $|C_i|$ ever reaches $16n_{1-i}/D_i < |V_{1-i}|/D_i$, one can establish a contradiction with the LE property by way of Lemma 3. Indeed, the set of

[1]Such graphs do not exist for all range of parameters, for instance, if $N_0 = 2^{N_1}$ such a strong expander needs $D_0 \ge c(\varepsilon)N_1$, which means that the graph needs to be very dense (see [9, Theorem 1.5(a)]).

non-free vertices would have to be larger than

$$|C_i|\, D_i/8 \geq (8n_{1-i}/D_i + |C_i|/2)D_i/8 = n_{1-i} + |C_i|\, D_i/16 > n_{1-i} + |C_i|\, \Delta_i$$

if we choose c sufficiently small and D_i/Δ_i sufficiently large.

Embedding scheme. Fix an arbitrary root $v_1 \in V_1(\mathcal{T})$ and map it to an arbitrary vertex in V_1'. At each step we take an already embedded vertex and embed all of its children at once. Suppose that we have a partial embedding f of $\mathcal{T}$ into G'. Let $\mathcal{C}$ be the collection of critical vertices and $\mathcal{R} = \{R_v\}_{v\in\mathcal{C}}$ be the family of reserved sets. Let $u \in V(\mathcal{T})$ be an embedded vertex and $w = f(u)$.

If w is critical then $R_w \in \mathcal{R}$ contains enough vertices to embed every child of u. No other critical vertex can be created after this embedding occurs (since no free vertex is used).

If $w \in V_i'$ is not critical, then the number of free neighbors of w is at least $cD_i \gg \Delta_i$, which is more than enough to embed every child of u. After embedding the children of u (arbitrarily choosing vertices among the free neighbors of w), we might have created new critical vertices.

A new critical vertex had cD_i free neighbors before the above embedding extension. Since the extension can only make Δ_i vertices non-free and $cD_i \gg \Delta_i$, this new critical vertex still has many free neighbors immediately after the extension.

Pick one of the (possibly many) new critical vertices and choose an arbitrary Δ_i-subset of its free neighborhood. We construct reserved sets for the new critical vertices using the following iterative procedure.

Suppose that $\mathcal{C}^j \subset V_i'$ is the collection of the first j critical vertices in V_i' created by the embedding extension. Let $\{R_v\}_{v\in\mathcal{C}^j}$ be a family of disjoint Δ_i-subsets such that each R_v may only contain free neighbors of v. Set $X^j = \bigcup_{v\in\mathcal{C}^j} R_v$.

If there is a (non-critical) vertex w having less than cD_i free neighbors outside of X^j we set $\mathcal{C}^{j+1} = \mathcal{C}^j \cup \{w\}$ and obtain a new family of disjoint Δ_i-sets $\{R_v\}_{v\in\mathcal{C}^{j+1}}$ as above. We also impose an extra restriction on this family: $X^j \subset X^{j+1}$, that is, once a vertex is chosen to be reserved to a critical vertex, it will be reserved to some critical vertex (but not necessarily to the one it was originally assigned to). This restriction is important since we use the fact that the set of non-free vertices is monotonically increasing. In particular, once a vertex is classified as critical, it always has less than cD_i free neighbors.

After the above procedure finishes, every non-critical vertex of V_i' $(i = 0, 1)$ has at least cD_i free neighbors and every critical vertex has an exclusive set of reserved vertices. Therefore, it is possible to continue the embedding until the whole tree is embedded.

Clearly, it is necessary to show that it is possible to obtain the above family of reserved sets. Each new critical vertex must have at least $cD_i - \Delta_i$ neighbors that are either free or temporarily reserved to another new critical vertex. Using the LE property of the graph and a Hall-type argument, it is simple to obtain a new family of reserved sets as long as $j = |\mathcal{C}^j|$ is not too large. However, since we have a global upper bound on the number of critical vertices, this strategy always work. (See Lemma 12 for a formal version of this argument.)

4. Properties of random bipartite graphs

In this section we state a technical theorem describing several properties of random bipartite graphs that we use when embedding trees. We remark that, in contrast with the assumptions of Section 3, having lossless expansion on both classes of a sparse bipartite graph is not always possible (see [9, Theorem 1.5(a)]). To overcome this shortcoming we show that there are plenty of LE sets in "useful places", namely, most neighborhoods of vertices are rich in LE sets.

Definition 5. Let $\varepsilon > 0$, $p \in (0, \varepsilon/8)$, $N_0, N_1, D_0 = pN_1, D_1 = pN_0 \in \mathbb{N}$. A bipartite graph $G = (U, W; E)$ with $|U| = N_0$, $|W| = N_1$ satisfies *Property* (‡) if there exists $W' \subseteq W$ with $|W'| \geq (1 - 2\varepsilon)N_1$ such that the following conditions hold:

(i) $\deg(w) \sim_\varepsilon D_1$ for all $w \in W'$ and, moreover, $\#\{u \in \Gamma(w) : \deg(u) \not\sim_\varepsilon D_0\} < \varepsilon D_1$;
(ii) for every $S \subseteq W'$ with $|S| \le \varepsilon N_1/(8D_0)$, we have $d^*(S) \sim_\varepsilon D_1 |S|$;
(iii) for every $S \subseteq W'$ with $|S| \le \varepsilon N_1/(D_0 D_1)$ and for every $T \subseteq \Gamma(S)$ with $\sqrt{\varepsilon} D_1 |S| \le |T|$, we have $d^*(T) \ge (1 - 5\sqrt{\varepsilon}) D_0 |T|$;
(iv) if $\varepsilon N_1 < D_0 D_1$ then for every $w \in W'$ and every subset $T \subseteq \Gamma(w)$ with $|T| \ge \varepsilon D_1$ we have disjoint sets $T_1, \ldots, T_r$, each of cardinality $\min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\}$, such that $\left|\bigcup_{i=1}^r T_i\right| \ge \frac{3}{4}|T|$ and $d^*(T_i') \sim_\varepsilon D_0 |T_i'|$ for every $T_i' \subseteq T_i$, $i = 1, \ldots, r$;
(v) for every $X \subseteq U$ and $Y \subseteq W$ with $|X| \ge \varepsilon^3 N_0$, $|Y| \ge \varepsilon^3 N_1$ we have $e_G(X, Y) \sim_{\varepsilon^2} p\,|X|\,|Y|$; in particular, $e(U, W') \ge (1 - 4\varepsilon)\, e(G)$.

Using the probabilistic method we show that there are graphs satisfying Property (‡).

Theorem 6. *Suppose that $n_0 \ge n_1$ and $n_0 \Delta_0 = n_1 \Delta_1$. Let $0 < \varepsilon < 1/100$ be given. There exists $C = C(\varepsilon)$ such that, with probability at least $1 - \varepsilon$, the bipartite random graph $G_{N_0,N_1;p} = (U, W; E)$, with $N_0 = Cn_0$, $N_1 = Cn_1$, and $p = \Delta_0/n_1 = \Delta_1/n_0 < \varepsilon/8$ satisfies **Property** (‡).*

Before proving the above theorem, we observe that the condition $p < \varepsilon/8$ is not very restrictive. In the case $p \ge \varepsilon/8$, we may use a complete bipartite graph.

Lemma 7. *Let $\alpha \in (0, 1]$ and $\mathcal{T}$ be a tree with (bipartite) classes having cardinalities n_0 and n_1. We have $G = K_{4n_0/\alpha, 4n_1/\alpha} \to_\alpha \mathcal{T}$.*

Proof. First observe that G has $16 n_0 n_1/\alpha^2$ edges. When $p \ge \varepsilon/8$, we must have $\beta(\mathcal{T}) \ge p n_0 n_1 \ge \varepsilon n_0 n_1/8$ and hence $e(G) = O(\beta(\mathcal{T}))$.

Let $G' \subseteq G$ be any subgraph with $e(G') \ge \alpha\, e(G)$. While there is a vertex v in the left class (or a vertex w in the right class) with $\deg_{G'}(v) < n_1$ (or $\deg_{G'}(w) < n_0$) remove v (or w) from G' together with all of the edges incident to the removed vertex. The total number of edges removed is upper bounded by $(4n_0/\alpha) n_1 + (4n_1/\alpha) n_0 = \frac{\alpha}{2} e(G)$. Therefore, the remaining graph G' is non-empty and has minimum degree on the left at least n_1 and minimum degree on the right at least n_0.

Now we can inductively embed any tree $\mathcal{T}$ with classes having cardinalities n_0 and n_1. Fix an arbitrary root $v_1 \in V_1(\mathcal{T})$ and set $f\colon v_1 \mapsto w_1$ where w_1 is an arbitrary vertex on the right class of G'.

Suppose that we have a partial embedding f of $\mathcal{T}$ into G. Pick some vertex $v \in V_i(\mathcal{T})$, $i = 0, 1$, that was already embedded together with some $w \in \Gamma_{\mathcal{T}}(v)$ which was not yet embedded. Since the degree of $f(v)$ in G' is at least as large as $|V_{1-i}(\mathcal{T})|$, there must be some $w' \in \Gamma_G(f(v))$ such that $w' \notin f(V_{1-i}(\mathcal{T}))$. Extend f by mapping w to w'. ■

To simplify the proof of Theorem 6 we shall avoid floors and ceilings by making every parameter—such as ε, p, C, n_0, n_1, Δ_0, Δ_1—a power of 2. This is not a problem given our final goal since this shall affect the parameter $\beta(\mathcal{T})$ by only a multiplicative constant.

Proof. The proof of Theorem 6 is divided into several claims.

Claim 8. *Let $G = G_{N_0,N_1;p} = (V_0, V_1; E)$ be a random bipartite graph and $S \subseteq V_i$ be a set with s vertices. Then $d^*(S)$ is a binomial variable with parameters N_{1-i} and $sp(1-p)^{s-1}$. Moreover, if $sp \le \varepsilon$ then $\mathbf{E}[d^*(S)] \ge (1 - 2\varepsilon) sp N_{1-i}$.*

Proof. We may represent $d^*(S)$ as a sum of indicator variables $I_v = \mathbb{I}[e_G(v, S) = 1]$, $v \in V_{1-i}$. Since the I_v's are independent and each has probability $sp(1-p)^{s-1}$, the first part of the claim is proved. For the second part, notice that

$$\mathbf{E}[d^*(S)] = N_{1-i} sp(1-p)^{s-1} \ge sp N_{1-i} e^{-2sp} \ge (1 - 2\varepsilon) sp N_{1-i}, \tag{3}$$

since $(1 - p) \ge e^{-p-p^2} \ge e^{-2p}$ (as $p \le \varepsilon/s \le 1/2$). □

Claim 9. *With probability at least* $1-3\varepsilon/4$ *there exists* $W' \subseteq W$ *with* $|W'| \geq (1-\varepsilon)N_1$ *for which* (i) *and* (ii) *hold.*

Proof. Notice that for any vertex v, we have $\mathbf{E}[\deg(v)] = D_i$, where $i = 0$ if $v \in U$ and $i = 1$ if $v \in W$. By the Chernoff inequality, for any fixed vertex v,

$$\mathbf{P}\left[\left|\deg(v) - D_i\right| \geq \varepsilon D_i\right] \leq \exp\{-\Omega_\varepsilon(D_i)\} \leq \varepsilon^2/4$$

for sufficiently large C.

Note that the degrees in W are independent random variables (since the graph is bipartite). Given a fixed vertex $w \in W$, let us estimate the probability that more than εD_1 of its neighbors have degree $\not\sim_\varepsilon D_0$ conditioned on $\deg(w) \sim_\varepsilon D_1$. For each $u \in \Gamma(w)$, the degree of u is one more than the number of its neighbors in $W - w$, which is a binomial variable independent of other vertices in $\Gamma(w)$ and of w itself. Hence, the probability of having εD_1 neighbors failing to have the "correct" degree is bounded by

$$\binom{(1+\varepsilon)D_1}{\varepsilon D_1} \exp\{-\Omega_\varepsilon(D_0) \cdot \varepsilon D_1\} = \exp\{-\Omega_\varepsilon(D_0 D_1)\} < \varepsilon^2/4,$$

for sufficiently large C.

Let $\mathcal{E}_0$ denote the event in which the set of vertices having exceptional degree or having many neighbors of exceptional degree has at most $\varepsilon N_1/2$ elements. Since the expected number of such vertices is less than $\varepsilon^2 N_1/4$, by Markov's inequality, we obtain $\mathbf{P}[\mathcal{E}_0] \geq 1 - \varepsilon/2$.

Next, we prove that the event

$$\mathcal{E}_1 = \left\{\text{for all } S \subseteq W \text{ with } s = |S| \in \left[\frac{\varepsilon}{8p}, \frac{\varepsilon}{4p}\right], \text{ we have } d^*(S) \geq (1-\varepsilon)D_1 |S|\right\}$$

holds with probability at least $1 - \varepsilon/4$. By Claim 8, we have $\mathbf{E}[d^*(S)] \geq (1-\varepsilon/2)sD_1$ for all sets considered in $\mathcal{E}_1$.

By the Chernoff inequality, the probability that one fixed set S in $\mathcal{E}_1$ has $d^*(S) < (1-\varepsilon)sD_1$ is at most by $\exp\{-\Omega_\varepsilon(sD_1)\}$. A simple union bound gives an upper bound on the probability that some set S has small $d^*(S)$, that is,

$$\sum_{s=\varepsilon/(8p)}^{\varepsilon/(4p)} \binom{N_1}{s} \exp\{-\Omega_\varepsilon(sD_1)\} \leq \sum_s \left\{\frac{eN_1 e^{-\Omega_\varepsilon(D_1)}}{s}\right\}^s \leq \sum_s \left\{\frac{8eD_0 e^{-\Omega_\varepsilon(D_1)}}{\varepsilon}\right\}^s.$$

Note that since $D_1 \geq D_0$, we may take C sufficiently large in order to have $e^{1-\Omega_\varepsilon(D_1)}D_0/\varepsilon < \varepsilon/64$. In particular, the last sum is at most $\sum_{s=1}^{\infty}(\varepsilon/8)^n < \varepsilon/4$.

To prove (ii) let us assume that $\mathcal{E}_1$ holds. Suppose that there are disjoint sets $S_1, S_2, \ldots, S_k$ such that $|S_i| \leq \varepsilon/(8p) - 1$ and $d^*(S_i) < (1-\varepsilon)D_1 |S_i|$. We call such sets S_i *non-expanding.* Suppose that $S = \bigcup_{i=1}^{k'} S_i$ ($k' \leq k$), is such that $\varepsilon/(8p) \leq |S| \leq 2(\varepsilon/(8p) - 1) \leq \varepsilon/(4p)$. Then $d^*(S) \leq \sum_{i=1}^{k'} d^*(S_i) < (1-\varepsilon)D_1 |S|$, which contradicts $\mathcal{E}_1$. It follows that by removing non-expanding sets from W sequentially we eventually get rid of all of them while removing at most $\varepsilon/(4p) = \varepsilon N_1/(4D_0)$ vertices.

In total, if both $\mathcal{E}_0$ and $\mathcal{E}_1$ hold, we need to remove less than εN_1 vertices from W to get (i) and (ii). Since $\mathbf{P}[\mathcal{E}_0 \wedge \mathcal{E}_1] \geq 1 - 3\varepsilon/4$ the claim is proved. □

Set $s_0 = \varepsilon N_1/(D_0 D_1)$. We assume that $s_0 \geq 1$ as otherwise (iii) is trivial. Let us estimate the probability that a fixed $S \subseteq W$ with $s = |S| \in [s_0, 3s_0]$ and $|\Gamma(S)| \sim_\varepsilon D_1 |S|$ is such that there exists $T \subseteq \Gamma(S)$ with $\varepsilon s D_1 \leq |T|$ having $d^*(T) < (1-10\varepsilon)D_0 |T|$. Such (S,T) will be called a *bad pair.* Apply Claim 8 to the random subgraph $G[U, W \setminus S]$ and the set T (observe that we have exposed the edges incident to S but no other edge of G, hence $G[U, W \setminus S]$ is a random graph independent of what was already exposed). Note that $p\,|T| \le (1+\varepsilon)D_1 sp \le 3(1+\varepsilon)\frac{\varepsilon N_1}{D_0}p = 3\varepsilon(1+\varepsilon)$ and $|W \setminus S| \ge (1-\varepsilon)N_1$. From Claim 8 we get that $\mathbf{E}[d^*(T)] \ge (1-8\varepsilon)D_0\,|T|$.

Applying the Chernoff inequality, we get that the probability that a fixed choice of (S,T) becomes a bad pair can be upper bounded by $\exp\{-\Omega_\varepsilon(D_0\,|T|)\}$. The union bound over all choices of S and all choices of T gives the following upper bound for the probability of any bad pair occurring in G:

$$\begin{aligned}[*] &= \sum_{s=s_0}^{3s_0}\sum_{t=\varepsilon sD_1}^{2sD_1}\binom{N_1}{s}\binom{2sD_1}{t}\exp\{-\Omega_\varepsilon(tD_0)\}\\ &\le \sum_{s=s_0}^{3s_0}\sum_{t=\varepsilon sD_1}^{2sD_1}\Big(\frac{eN_1}{s}\Big)^s\Big(\frac{2sD_1}{t}\Big)^t\exp\{-\Omega_\varepsilon(tD_0)\}.\end{aligned}$$

Replacing the occurrences of s and t in the denominators by lower bounds (s_0 and $\varepsilon s_0 D_1$, respectively) and their occurrences in the numerators or exponents by upper bounds ($3s_0$ and $6s_0D_1$, respectively) we obtain

$$\begin{aligned}[*] &\le \sum_{t=\varepsilon s_0 D_1}^{6s_0D_1}\sum_s (2eD_0D_1/\varepsilon)^{3s_0}(6e/\varepsilon)^t\exp\{-\Omega_\varepsilon(tD_0)\}\\ &\le \sum_t 2s_0\cdot\exp\big\{3s_0\log(2eD_0D_1/\varepsilon)+t\log(6e/\varepsilon)-\Omega_\varepsilon(tD_0)\big\}\\ &\le 12s_0^2D_1\cdot\exp\big\{3s_0\log(2eD_0D_1/\varepsilon)+6s_0D_1\log(6e/\varepsilon)-\Omega_\varepsilon(s_0D_0D_1)\big\}\\ &\le \exp\{-\Omega_\varepsilon(N_1)\},\end{aligned} \tag{4}$$

for a sufficiently large C.

Let $\mathcal{E}_2$ be the event

$$\begin{aligned}\mathcal{E}_2 \equiv \Big\{&\text{for all } S\subseteq W, \text{ with } s=|S|\in[s_0,3s_0] \text{ and } |\Gamma(S)|\sim_\varepsilon sD_1,\\ &\text{if } T\subseteq\Gamma(S),\ \varepsilon sD_1\le|T|, \text{ then } d^*(T)\ge(1-10\varepsilon)D_0\,|T|\Big\}.\end{aligned} \tag{5}$$

By equation (4), $\mathcal{E}_2$ holds with probability at least $1-\varepsilon/16$.

Claim 10. *Conditioning on $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$, there is $W'\subseteq W$ satisfying* (i), (ii), (iii).

Proof. Initially, let W' be the set obtained by Claim 9 (here we use $\mathcal{E}_0$ and $\mathcal{E}_1$). Suppose that there exists $S_1\subseteq W'$ with $|S_1|\le s_0-1$ and $\Gamma_G(S_1)\sim_\varepsilon D_1\,|S_1|$ such that there is $T_1\subseteq\Gamma_G(S_1)$ with $\sqrt{\varepsilon}D_1\,|S_1|\le|T_1|$ and $d^*(T_1)<(1-5\sqrt{\varepsilon})D_0\,|T_1|$. Remove S_1 from W'. Repeat this procedure until there are no more bad sets or until the union $S=\bigcup_i S_i$ has at least s_0 elements. If there is a (first) step k in which $|S|=\sum_{i=1}^k|S_i|\ge s_0$, we also have $|S|\le 2s_0$ since we are always adding sets with less than s_0 elements. Next we show that S cannot have more than s_0 elements.

Suppose that $s_0\le|S|\le 2s_0$ and let $T=\bigcup_{i=1}^k T_i\subseteq\Gamma_G(S)$. Exploiting the LE property of W' we shall show that $|T|$ is close to $\sum_{i=1}^k|T_i|$ and, since $T\subseteq\Gamma(S)$, this contradicts $\mathcal{E}_2$. Note that $e_G(S,T)\ge\sum_{i=1}^k e_G(S_i,T_i)$, since the S_i's are disjoint. However, we know that $e_G(S_i,T_i)\ge|T_i|\ge\sqrt{\varepsilon}D_1|S_i|$. Take $G'\subseteq G$ with $E(G')=\bigcup_{i=1}^k e_G(S_i,T_i)$. Clearly, $e(G')\ge\sum_{i=1}^k|T_i|\ge\sqrt{\varepsilon}D_1\,|S|$. On the other hand, since W' was initially obtained from Claim 9, every vertex of W' has degree at most $(1+\varepsilon)D_1$ and $|\Gamma_G(S)|\ge d_G^*(S)\ge(1-\varepsilon)D_1\,|S|$. Hence, by Lemma 3, it follows that

$$\begin{aligned}|T|=|\Gamma_{G'}(S)| &\ge e_{G'}(S,T)-2\{e_G(S,T)-|\Gamma_G(S)|\}\\ &\ge\sum_{i=1}^k|T_i|-4\varepsilon D_1\,|S|\ge(1-4\sqrt{\varepsilon})\sum_{i=1}^k|T_i|,\end{aligned} \tag{6}$$

where we have used that $\sqrt{\varepsilon}D_1\,|S| \le \sum_{i=1}^k |T_i|$. Therefore $|T| \ge \frac{1}{2}\sum_{i=1}^k |T_i| \ge \frac{1}{2}\sqrt{\varepsilon}D_1\,|S| > \varepsilon D_1\,|S|$.

Since, $\Gamma^*_G(T) \subseteq \bigcup_{i=1}^k \Gamma^*_G(T_i)$ and $\varepsilon < 1/100$, we have from (6),

$$\begin{aligned} d^*_G(T) &\le \sum_{i=1}^k d^*_G(T_i) < (1-5\sqrt{\varepsilon})D_0 \sum_{i=1}^k |T_i| \\ &\le \frac{1-5\sqrt{\varepsilon}}{1-4\sqrt{\varepsilon}} D_0\,|T| \\ &< (1-10\varepsilon)D_0\,|T|, \end{aligned}$$

a contradiction with $\mathcal{E}_2$. Hence, by removing less than s_0 elements from W' we may ensure that (iii) holds together with (i) and (ii). □

Claim 11. *If* $\varepsilon N_1 < D_0 D_1$ *then* ***a.a.s.*** *every* $w \in W$ *for which* $\deg(w) \sim_\varepsilon D_1$ *and every* $T \subseteq \Gamma(w)$ *with* $|T| \ge \varepsilon D_1$ *satisfy the conditions of Property (‡).(iv).*

Proof. Suppose that $\varepsilon N_1 < D_0 D_1$. Let $w \in W$ be fixed and assume that $\deg(w) \sim_\varepsilon D_1$ (as otherwise $w \notin W'$). Let $T = \{t_1, t_2, \ldots, t_m\} \subseteq \Gamma(w)$ be an arbitrary set with $m \ge \varepsilon D_1$. In the random graph $G[U, W \setminus \{w\}]$, the vertex t_1 has expected degree $p(N_1 - 1) \sim_{\varepsilon/2} D_0$. Hence, by the Chernoff inequality,

$$\mathbf{P}[\deg(t_1) \sim_\varepsilon D_0] \ge 1 - \exp\{\Omega_\varepsilon(D_0)\}.$$

We shall (attempt to) construct a set T_1 with $k = \min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\}$ elements satisfying condition (iv). Let $X = \{w\}$. We say that t_i *succeeds* if $|\Gamma(t_i) \setminus X| \sim_\varepsilon D_0$ and $\deg(t_i) \sim_\varepsilon D_0$, otherwise it *fails*. If t_i succeeds, we add t_i to T_1 and $\Gamma(t_i)$ to X. If it fails, both X and T_1 remain unchanged. If T_1 contains k elements then we have obtained our final T_1. By construction, every $T_1' \subseteq T_1$ is such that $d^*(T_1') \sim_\varepsilon D_0\,|T_1'|$.

Suppose that t_ℓ was the last element added to T_1. Then we start building $T_2 \subset \{t_{\ell+1}, \ldots, t_m\}$ in the same way we constructed T_1: set $X = \{w\}$ and sequentially add vertices t_i that succeed to T_2 and their neighborhoods $\Gamma(t_i)$ to X. Repeat the procedure for other T_i's until we have constructed T_r or the vertex t_m was reached. Note that we always have $|X| \le \frac{\varepsilon N_1}{4D_0}(1+\varepsilon)D_0 + 1 \le \varepsilon N_1/2$. Therefore, by the Chernoff inequality, the probability that a fixed t_i fails is at most $\exp\{-\Omega_\varepsilon(D_0)\}$ for any i.

If we were unable to construct the desired collection $T_1, \ldots, T_r$ then at least $m/8$ elements from T have failed. Indeed, we need rk elements to succeed, where $3m/4 \le rk < 3m/4 + k \le 3m/4 + \varepsilon D_1/8 \le 7m/8$. Even though the sequence of indicator variables $\mathbb{I}[t_i \text{ fails}]$ is not independent, it is dominated by a sequence of independent Bernoulli variables. Therefore, the probability that a fixed sequence of $(m/8) \ge \varepsilon D_1/8$ vertices fails is at most $\exp\{-\Omega_\varepsilon(mD_0/8)\} = \exp\{-\Omega_\varepsilon(D_0 D_1\}$. Consider the union bound over (1) all choices $w \in W$ having $\deg(w) \sim_\varepsilon D_1$; (2) all subsets $T \subseteq \Gamma(w)$ with $|T| \ge \varepsilon D_1$; (3) all possible $(m/8)$-subsets of failing vertices of T. The probability that we fail to construct the desired collection for some vertex is at most

$$N_1 \cdot 2^{2D_1} \cdot 2^{2D_1} \cdot \exp\{-\Omega_\varepsilon(D_0 D_1)\} \to 0 \text{ as } D_0 D_1 \to \infty, \tag{7}$$

since $D_0 D_1 > \varepsilon N_1$. (Note that we can make $D_0 D_1$ as large as needed by taking a sufficiently large C.) Thus the claim is proved. □

It is a well-known fact that the number of edges among linear-sized sets in a random graph is **a.a.s.** very close to the expected value. Indeed, let $\mathcal{E}_3$ be the event corresponding to (‡).(v) and let $\mathcal{E}_4$ denote the event described by Claim 11. Note that the events $\mathcal{E}_0, \ldots, \mathcal{E}_4$ hold together with probability at least $1 - \varepsilon$. Conditioning on all those events, (v) is satisfied (by $\mathcal{E}_3$), Claim 10 ensures (i)–(iii) and $\mathcal{E}_4$ together with (i) imply (iv). ■

5. Auxiliary results

In this section we prove lemmas that will be used to ensure that certain steps in our tree embedding scheme can be performed.

Lemma 12. *Let $S_1, \dots, S_m$ be a collection of sets and $b \in \mathbb{N}^m$ be such that, for every $I \subseteq [m]$, we have $\left|\bigcup_{i\in I} S_i\right| \geq \sum_{i\in I} b_i$.*

Then, there exists a disjoint family $\mathcal{S} = \{S'_i \subseteq S_i\}_{i=1}^m$ with $|S_i| = b_i$ for all i. Moreover, if $\{S''_i \subseteq S_i\}_{i=1}^k$, $k \leq m$, is any disjoint family with $|S''_i| = b_i$, we may find $\mathcal{S}$ such that $\bigcup_{i=1}^k S''_i \subseteq \bigcup_{i=1}^m S'_i$.

Proof. We reduce this problem to a matching problem. Consider a bipartite graph H with vertex classes $A = \bigcup_{i=1}^m \{i\} \times [b_i]$ and $B = \bigcup_{i=1}^m S_i$ and edges given by $\{(i,j), u\}$ for all $i \in [m]$, $j \in [b_i]$ and $u \in S_i$. Observe that we are adding b_i copies of a vertex i that has neighborhood S_i for all i.

Given a set $A' \subseteq A$, let $I = I(A')$ be the projection of A' onto the first coordinate. We have $|A'| \leq \sum_{i\in I} b_i$ and, on the other hand, $|\Gamma_H(A')| = \left|\bigcup_{i\in I} S_i\right| \geq \sum_{i\in I} b_i \geq |A'|$. Hence, Hall's condition is satisfied for H and there is a matching M covering A. From M we get sets $S'_i \subseteq S_i$ by letting S'_i be the set of elements matched to $(i,1), \dots, (i, b_i)$.

Suppose that there exists a disjoint family $\{S''_i \subseteq S_i\}_{i=1}^k$, $k \leq m$, with $|S''_i| = b_i$. By performing small local changes to the family $\{S'_i \subseteq S_i\}_{i=1}^m$ we may ensure that $\bigcup_{i=1}^k S''_i \subseteq \bigcup_{i=1}^m S'_i$. If there exists $x \in \bigcup_{i=1}^k S''_i \setminus \bigcup_{i=1}^m S'_i$ then let $j \in [k]$ be such that $x \in S''_j$. Since $b_j = |S'_j| = |S''_j|$, there exists some $y \in S'_j \setminus S''_j$. Set $S'_j \leftarrow S'_j - y + x$. Note that this strictly decreases

$$\sum_{i=1}^k |S'_i \triangle S''_i|.$$

In particular, since this number is always non-negative, in at most $\sum_{i=1}^k |S'_i \triangle S''_i|$ steps we can obtain the desired family. ■

Lemma 13. *Let $G = (U, W; E)$ be a graph with $W' \subseteq W$ satisfying Property (‡). Let $\alpha \geq \alpha_0(\varepsilon) = \Omega(\sqrt{\varepsilon})$.*

Suppose that $S \subseteq W'$, with $|S| \leq \varepsilon N_1/(D_0 D_1)$, is such that there is a disjoint family $\{A_v \subset \Gamma(v)\}_{v\in S}$ and a (not necessarily disjoint) family $\{B_x \subset \Gamma(x)\}_{x\in\bigcup_{v\in S} A_v}$ with $|A_v| = \alpha D_1$ for every $v \in S$ and $|B_x| = \alpha D_0$ for every $x \in \bigcup_{v\in S} A_v$.

Then there is a disjoint family of Δ_1-sets $\{X_v \subseteq A_v\}_{v\in S}$ and a disjoint family of Δ_0-sets $\{Y_{v,x} \subseteq B_x\}_{v\in S, x\in X_v}$.

Proof. We shall assume that $D_0 D_1 \leq \varepsilon N_1$ as otherwise $S = \emptyset$ and there is nothing to prove. The desired families will be obtained in three steps.

In step one we obtain a disjoint family $\{X'_v \subseteq A_v\}_{v\in S}$ such that, for every $v \in S$, we have that $|X'_v| = m = (\alpha - O(\sqrt{\varepsilon}))D_1$ and every $u \in X'_v$ has $\deg(u) \sim_\varepsilon D_0$. This is done by deleting at most εD_1 vertices from A_v (see Property (‡).(i)).

In step two we obtain a disjoint family

$$\Big\{Y'_v \subseteq Y_v = \bigcup_{x\in X'_v} B_x\Big\}_{v\in S}$$

with $|Y'_v| = (\alpha - O(\sqrt{\varepsilon}))D_0\,|X'_v|$. For $S' \subseteq S$, denote by $X_{S'}$ the union $X_{S'} = \bigcup_{v\in S'} X'_v$. Note that we have $X_{S'} \subseteq \Gamma(S')$ with $|X_{S'}| = m\,|S'| \geq \sqrt{\varepsilon} D_1\,|S'|$. Hence, from Property (‡).(iii) we get that $|\Gamma(X_{S'})| \geq d^*(X_{S'}) \geq (1 - 5\sqrt{\varepsilon})D_0\,|X_{S'}|$. Using the degree hypothesis on the elements of the set X'_v and applying Lemma 3 we conclude that

$$\Big|\bigcup_{v\in S'} Y_v\Big| \geq \#\big\{\{x,y\}\in E(G) \,:\, x\in X_{S'}, y\in B_x\big\} - 2\{\Gamma(X_{S'}) - e(X_{S'},W)\}$$
$$\geq (\alpha - O(\sqrt{\varepsilon}))D_0\,|X_{S'}| = (\alpha - O(\sqrt{\varepsilon}))D_0 m\,|S'|.$$

Using Lemma 12 we may obtain the desired family of disjoint sets $Y'_v \subseteq Y_v$ with $|Y'_v| = (\alpha - O(\sqrt{\varepsilon}))D_0 m$ for all $v\in S$.

In step three we obtain the families described in the statement of this lemma.

Consider the pair (X'_v, Y'_v) for some $v\in S$. Let us construct a Δ_1-set $X_v \subset X'_v$ in the following way. Initially, set $X_v \leftarrow \emptyset$. Suppose that we have a disjoint family of Δ_0-sets $\{Y_{v,x}\subset Y'_v\}_{x\in X_v}$. While $|X_v| < \Delta_1$ and there exists $x\in X'_v\setminus X_v$ such that $Z_x = (B_x\cap Y'_v)\setminus\bigcup_{u\in X_v} Y_{v,u}$ satisfies $|Z_x|\geq \Delta_0$, add x to X_v and take an arbitrary Δ_0-subset $Y_{v,x}\subseteq Z_x$.

Let $Y''_v = Y'_v\setminus\bigcup_{x\in X_v} B_x$ and suppose that $|X_v| < \Delta_1$. Notice that we have $|Y''_v| \geq (\alpha - O(\sqrt{\varepsilon}))D_0 m - \Delta_1\alpha\, D_0 > \Delta_0 m$. Since $Y''_v\subset\bigcup_{x\in X'_v\setminus X_v} B_x$, it follows that

$$\sum_{x\in X'_v\setminus X_v} |B_x\cap Y''_v| \geq |Y''_v| > \Delta_0 m$$

and hence there must be some $x\in X'_v\setminus X_v$ with $|B_x\cap Y''_v| > \Delta_0$. Because $Y_{v,u}\subset B_u$ for all $u\in X_v$ we have $Z_x\supseteq B_x\cap Y''_v$ and thus x can be added to X_v. As the sets $\{Y'_v\}_{v\in S}$ are pairwise disjoint, so are the sets $\{Y_{v,x}\subseteq B_x\cap Y'_v\}_{v\in S, x\in X_v}$. ■

6. An embedding scheme for trees

In this section we present Algorithm 1, which embeds trees in suitable graphs. This algorithm takes advantage of the lossless expansion property of the host graph when constructing the embedding. Although many of the techniques and ideas involved in this algorithm were already discussed in a superficial level in Section 3, there are many new details and subtleties that are addressed solely in this section.

A formal analysis of the algorithm is done through several invariants that must be true at the beginning of every iteration. Once the invariants are known to hold at the beginning of every iteration, we must prove that the algorithm does not abort. If the algorithm does not abort then it succeeds in embedding the tree, which is our goal.

In what follows, α will be a fixed number and $r_C, r_D \in \mathbb{N}$ will be sufficiently large absolute constants.

Invariant 14. At the beginning of every iteration of Algorithm 1 (line 1.10), the following holds:

I. **(cardinality of $|Z|$)**, we have

$$|Z\cap U| \leq |f_M(\mathcal{T})\cap U| + |\mathcal{D}| + |\mathcal{C}|(\alpha 2^{-r_C} D_1)$$

and

$$|Z\cap W| \leq |f_M(\mathcal{T})\cap W| + |\mathcal{U}|(\alpha^2 2^{-r_C - r_U} D_0 D_1);$$

II. **(non-critical/non-dangerous vertices)** for every $u\in U\setminus\mathcal{D}$, $w\in W\setminus\mathcal{C}$, we have

$$\deg_G(u, W\setminus Z) \geq \frac{\alpha D_0}{2} \quad\text{and}\quad \deg_G(w, U\setminus Z) \geq \frac{\alpha D_1}{2};$$

III. **(dangerous vertices)** we have $|\mathcal{D}| < \varepsilon^3 N_0$ and, for every $u\in\mathcal{D}\subseteq U$,

$$\deg_G(u, Z\cap W) \geq \frac{\alpha D_0}{2}.$$

IV. **(critical vertices)** for every $w \in \mathcal{C} \subseteq W$ we have

$$\deg_G(w, Z \cap U) \geq \frac{\alpha D_1}{2},$$

and the set $S_w \in \mathcal{S}$ has $\alpha 2^{-r_C} D_1$ elements exclusively reserved for embedding the children of w; moreover, if $w \notin \mathcal{U}$, then

$$|S_w \setminus \mathcal{D}| = \#\{u \in S_w \ : \ \deg_G(u, W \setminus Z) \geq \alpha D_0/2\} \geq |S_w|/2 = \alpha 2^{-1-r_C} D_1; \tag{8}$$

V. **(ultra-critical vertices)** for every $w \in \mathcal{U} \subseteq \mathcal{C}$,

$$\#\{u \in S_w \ : \ \deg_G(u, W \setminus Z) < \alpha D_0/2\} > |S_w|/2 = \alpha 2^{-1-r_C} D_1; \tag{9}$$

moreover, we also have $S''_w \in \mathcal{S}_U$ with $|S''_w| = \Delta_1$ and a family of Δ_0-sets $\{Z_{w,u}\}_{u \in S''_w} \subseteq \mathcal{W}$, where S''_w is reserved for children of w and $Z_{w,u}$ is reserved for children of u (grandchildren of w).

Algorithm 1: Embedding trees

Input : A tree $\mathcal{T}$ with root $r \in V_1(\mathcal{T})$;
A graph $G = (U, W; E)$.
Output: An embedding of $\mathcal{T}$ into G represented by a matching M.

1.1 $M \leftarrow \{(r, \min(W))\}$; `// initialize embedding`
1.2 $Q \leftarrow \{\min(W)\}$; `// queue of active vertices`
1.3 $\mathcal{C} \leftarrow \emptyset$; `// critical vertices`
1.4 $\mathcal{D} \leftarrow \emptyset$; `// dangerous vertices of` U
1.5 $\mathcal{S} \leftarrow \emptyset$; `// reserved neighborhoods (family of subsets of` U`,` $\mathcal{S} = \{S_v\}_{v \in \mathcal{C}}$`)`
1.6 $\mathcal{U} \leftarrow \emptyset$; `// ultra-critical vertices`
1.7 $\mathcal{S}_U \leftarrow \emptyset$; `// reserved neighborhoods for children of ultra-critical vertices`
1.8 $\mathcal{W} \leftarrow \emptyset$; `// reserved neighborhoods for grandchildren of ultra-critical vertices`
1.9 $Z \leftarrow \{1\}$; `// set of used, reserved or dangerous vertices`
1.10 **while** $Q \neq \emptyset$ **do**
1.11 $\quad p \leftarrow$ **pop** (Q) ; `// get an active vertex`
1.12 $\quad$ **if** $p \in \mathcal{U}$ **then**
1.13 $\quad\quad (M, S''_p, \{Z'_{p,u}\}_{u \in S''_p}) \leftarrow$ **embed-descendants** $(M, p, S'_p, \{Z_{p,u}\}_{u \in S'_p})$; `//` $S'_p \in \mathcal{S}_U$ `and` $Z_{p,u} \in \mathcal{W}$
1.14 $\quad\quad$ **enqueue** $\left(Q, \bigcup_{u \in S''_p} Z'_{p,u}\right)$;
1.15 $\quad\quad$ **go-to** 1.10 ; `// skip to the next iteration`
1.16 $\quad C_p \leftarrow \{v_1, \ldots, v_l \ : \ v_i$ is a child of $f_M^{-1}(p)\}$;
1.17 $\quad$ **if** $p \notin \mathcal{C}$ **then**
1.18 $\quad\quad S_p \leftarrow \Gamma_G(p) \setminus Z$; `// if` $p \in \mathcal{C}$ `then` $S_p \in \mathcal{S}$ `is already defined`
1.19 $\quad$ find a subset $S'_p = \{u_1, \ldots, u_l\} \subseteq S_p$ and a disjoint family $\{Z_i \subseteq \Gamma_G(u_i) \setminus Z\}_{u_i \in S'_p}$, with $|Z_i| = \#\{$children of $v_i\}$; if not possible, **abort** ;
1.20 $\quad$ extend M: match v_i to u_i and $\{$children of $v_i\}$ to Z_i arbitrarily for all i ;
1.21 $\quad$ **enqueue** $\left(Q, \bigcup_{i=1}^{l} Z_i\right)$;
1.22 $\quad Z \leftarrow Z \cup S'_p \cup \bigcup_{i=1}^{l} Z_i$;
1.23 $\quad$ **restore-invariants**;

Procedure `embed-descendants(`$M, p, S_p, \{Z_{p,u}\}_{u\in S_p}$`)`

Input : M – current embedding, $f = f_M$ is the corresponding function;
p – a vertex in the host graph already used in the embedding;
S_p – children of p should be mapped into this set;
$\{Z_{p,u}\}_{u\in S_p}$ – if a child v of $f^{-1}(p)$ is mapped to $u \in S_p$, the children of v will be mapped into $Z_{p,u}$.

Output: M – updated embedding;
$S'_p \subseteq S_p$ – vertices used for children of $f^{-1}(p)$;
$\{Z'_{p,u} \subseteq Z_{p,u}\}_{u\in S'_p}$ – vertices used for grandchildren of $f^{-1}(p)$.

```
choose S'_p ⊆ S_p arbitrarily with |S'_p| = deg_T(f^{-1}(p)) ;
match each v ∈ Γ_T(f^{-1}(p)) to some vertex in S'_p and update M ;
for each u ∈ S'_p, take arbitrarily some Z'_{p,u} ⊆ Z_{p,u} with |Z'_{p,u}| = deg_T(f^{-1}(u)) ;
for each u ∈ S'_p and each w ∈ Γ_T(f^{-1}(u)), match w to a vertex in Z'_{p,u} and update M ;
```

Procedure `restore-invariants`

```
R ← ∅ ;
D ← {x ∈ U : deg_G(x, W \ Z) < αD_0/2} ;
Z' ← ∅ ;
repeat
    (C', S') ← find-critical-vertices (Z, C, D) ;
    // consolidate critical vertices
    C ← C ∪ C', S ← S ∪ S', Z ← Z ∪ ⋃_{S∈S'} S ;
    // promotion to ultra-critical
    U' ← {w ∈ C \ U : |S_w \ D| < |S_w|/2 = α2^{-1-r_C} D_1} ;
    U ← U ∪ U' ;
    if |U| > εN_1/(D_0 D_1) then
        abort Algorithm 1 ;
    find sets S'_w ⊆ S_w with |S'_w| = |S_w|/4, for w ∈ U', and a (not necessarily disjoint) family of
    α2^{-r_U} D_0-sets {Y_{w,u} ⊆ Γ_G(u) \ Z}_{w∈U', u∈S'_w}; if not possible, abort Algorithm 1 ;
    Z' ← Z' ∪ ⋃_{w∈U', u∈S'_w} Y_{w,u} ;
    R ← R ∪ U'
    D ← {x ∈ U : deg_G(x, W \ (Z ∪ Z')) < αD_0/2}
until U' = ∅ ;
find sets S''_w ⊆ S'_w with |S''_w| = Δ_1, for w ∈ R, and a disjoint family of Δ_0-sets
{Z_{w,u} ⊆ Y_{w,u}}_{w∈R, u∈S''_w}; if not possible, abort Algorithm 1 ;
D ← D, Z ← Z ∪ Z' ∪ D ;
S_U ← S_U ∪ {S''_w}_{w∈R} ;
W ← W ∪ {Z_{w,u}}_{w∈R, u∈S''_w} ;
```

Theorem 15. *Let $n_0, n_1, \Delta_0, \Delta_1$ be given. Suppose that $G' = (U, W; E)$ is a graph satisfying Property (‡) for some $\varepsilon > 0$, $N_0 = Cn_0$, $N_1 = Cn_1$ (with C sufficiently large) and $p = \max\{\Delta_0/n_1, \Delta_1/n_0\} < \varepsilon/8$. Let $W' \subset W$ be determined by Property (‡), $D_0 = pN_1$ and $D_1 = pN_0$.*

There exists an absolute constant $c > 0$ such that the following holds. Let $\alpha = c\sqrt{\varepsilon}$ and $G \subseteq G'[U, W']$ be such that $d_G(u) \geq \alpha D_0$ for all $u \in U \cap V(G)$ and $d_G(w) \geq \alpha D_1$ for all $w \in W' \cap V(G)$. Then, Algorithm 1 embeds any $(n_0, \Delta_0, n_1, \Delta_1)$-tree $\mathcal{T}$ into G.

Procedure `find-critical-vertices`$(Z, \mathcal{C}, D)$

Input : Z – set of used/reserved/dangerous vertices;
$\mathcal{C}$ – current collection of critical vertices;
D – set of vertices that will be marked dangerous.

Output: $\mathcal{C}'$ – the set of critical vertices found;
$\{S_w \subseteq \Gamma_G(w) \setminus Z\}_{w \in \mathcal{C}'}$ – a disjoint family of $\alpha 2^{-r_C} D_1$-sets.

4.1 $\mathcal{C}' \leftarrow \emptyset$
4.2 $X \leftarrow \emptyset$
4.3 **while** *there exists* $v \in W \setminus (\mathcal{C} \cup \mathcal{C}')$ *with* $\deg_G(v, U \setminus (Z \cup X \cup D)) < \alpha D_1/2$ **do**
4.4 $\quad \mathcal{C}' \leftarrow \mathcal{C}' + v$
4.5 $\quad$ **if** $|\mathcal{C}'| + |\mathcal{C}| > \varepsilon N_0/(4D_1)$ **then**
4.6 $\quad\quad$ **abort** Algorithm 1 ;
4.7 $\quad$ find disjoint family of $\alpha 2^{-r_C} D_1$-sets $\{S_w \subseteq \Gamma_G(w) \setminus Z\}_{w \in \mathcal{C}'}$ covering X; if not possible, **abort** Algorithm 1 ;
4.8 $\quad X \leftarrow \bigcup_{w \in \mathcal{C}'} S_w$;
4.9 **return** $(\mathcal{C}', \{S_w\}_{w \in \mathcal{C}'})$

Proof. We shall abuse the notation and set $U \leftarrow U \cap V(G)$ and $W \leftarrow W' \cap V(G)$. Hence G has classes U and W. Moreover, we shall assume that $W = \{1, 2, \ldots, |W|\}$. The proof is divided into two parts:

- The Invariants I-V hold at the beginning of every iteration;
- Algorithm 1 does not abort when the input consists of an $(n_0, \Delta_0, n_1, \Delta_1)$-tree $\mathcal{T}$ (with arbitrary root $r \in V_1(\mathcal{T})$) and G as above.

For the base case, we have $Z = \{1\} = Z \cap W = f_M(\mathcal{T})$ and there are no critical or dangerous vertices. It is then easy to check that all the invariants hold.

Next, observe that when $p \in \mathcal{U}$, the sets Z, $\mathcal{C}$, $\mathcal{U}$, $\mathcal{S}$, $\mathcal{W}$ remain unchanged. Also, the elements $S''_p \subseteq S'_p \in \mathcal{S}_U$ are used for the children of p and the elements of $\{Z_{p,u}\}_{u \in S''_p} \subseteq \mathcal{W}$ are used for the grandchildren of p. Hence, in this case, the invariants are maintained and the algorithm does not abort.

Suppose that all the invariants hold at the beginning of some iteration and that $p \notin \mathcal{U}$. Let us prove that all the invariants hold at the beginning of the next iteration (if the algorithm does not abort).

Proof of Invariant I. Examining the steps where Z is updated (see lines 1.9, 1.22, 3.17), it is clear that, by the end of the iteration, $Z \cap U$ consists of vertices used by the embedding $(f_M(\mathcal{T}) \cap U)$, dangerous vertices (namely, $\mathcal{D}$) and reserved vertices $(\cup_{w \in \mathcal{C}} S_w)$ which account for $|C|(\alpha 2^{-r_C} D_1)$ vertices. It is also clear that $Z \cap W$ contains vertices used in the embedding $(f_M(\mathcal{T}) \cap W)$ and the vertices added to Z' by line 3.12, which are upper bounded by $\sum_{w \in \mathcal{U}} |S_w| \, \alpha 2^{-r_U} D_0$. The invariant follows. □

Proof of Invariant II. Let us analyze the Procedure `restore-invariants`. By construction (see line 4.3), immediately after Procedure `find-critical-vertices` returns (line 3.5) and the critical vertices are consolidated (in particular, Z now contains the newly reserved neighborhoods), no vertex $w \in W \setminus \mathcal{C}$ satisfies $\deg_G(w, U \setminus (Z \cup D)) \leq \alpha D_1/2$.

If $\mathcal{U}'$ is empty on the beginning of some iteration of the inner loop, the loop will be complete at that iteration without changing D or Z any further. In particular, the degree condition for non-critical vertices $(W \setminus \mathcal{C})$ is ensured at the end of the iteration and this part of Invariant II holds at the next iteration.

The case of $u \in U \setminus \mathcal{D}$ is simpler: any vertex that does not satisfy the degree condition by the end of the iteration is either already dangerous or becomes dangerous (see lines 3.2 and 3.14). ☐

Proof of Invariant III. The degree part of Invariant III follows immediately from the updates made to $\mathcal{D}$ (lines 3.2 and 3.14) and the fact that Z never loses any element. The fact that $\mathcal{D} \subset Z$ easily follows from line 3.17.

It remains to upper bound the number of dangerous vertices. Observe that $Z \cap W$ only contains reserved neighbors and embedded vertices and its cardinality is determined by Invariant I (which is already proven to hold at the next iteration). Also note that a dangerous vertex v must have, by the end of the iteration, $\deg_G(v, Z \cap W) \geq \alpha D_0/2$. We now use Property(‡).(v), to derive a bound on $\mathcal{D}$. We have

$$|Z \cap W| \leq n_1 + 2\frac{\varepsilon N_1}{D_0 D_1}\alpha^2 2^{-r_C - r_U} D_0 D_1 \leq \varepsilon N_1.$$

On the other hand,

$$e_{G'}(\mathcal{D}, Z \cap W) \geq |\mathcal{D}|(\alpha D_0/2) = p\,|\mathcal{D}|\Big(\frac{\alpha}{2}N_1\Big).$$

Hence, if $|\mathcal{D}| \geq \varepsilon^3 N_0$ then $Z \cap W$ should have at least $\alpha N_1/4 > \varepsilon N_1$ vertices, a contradiction. ☐

Proof of Invariant IV. There is only one place in the algorithm where the set of critical vertices grows—just after a call to Procedure `find-critical-vertices` (l. 3.5) these critical vertices are consolidated. A subtle, but very important detail of `find-critical-vertices` consists in requiring that the family obtained in line 4.7 covers the set X (which is the union of the reserved sets of the previous iteration). Hence, a vertex that had a small number of free neighbors could only have less free neighbors in following iterations.

We also note that once an element is added to D, it remains in D (and is subsequently added to $\mathcal{D}$). It is immediate that the number of edges a critical vertex sends into Z cannot become smaller than $\alpha D_1/2$.

The reserved neighborhoods are defined to have $\alpha 2^{-r_C} D_1$ elements each and, once a reserved neighborhood is finally determined (after Procedure `find-critical-vertices` returns), it is consolidated by being merged into Z. Since the reserved neighborhoods are disjoint, and new reserved neighborhoods must be chosen outside of Z, no other vertex can have its children embedded in a reserved neighborhood.

Moreover, if a vertex $w \in \mathcal{C}$ fails equation (8) at the end of the iteration, the line 3.7, together with the condition of the inner loop (that $\mathcal{U}' = \emptyset$), ensures that $w \in \mathcal{U}$ will hold when the iteration ends. ☐

Proof of Invariant V. If no new ultra-critical vertex was found at the iteration, the invariant is preserved. Hence, let us assume that some ultra-critical vertex was found.

Following the construction of $\mathcal{U}'$ (see line 3.7), at the moment a vertex w becomes ultra-critical, equation (9) holds. Since the set Z is monotonically increasing, this equation must continue to hold subsequently.

The family of reserved sets of Invariant V is obtained at line 3.16. Those reserved vertices will not be used to embed the children/grandchildren of any other vertex because the reserved sets are merged into Z and no other vertex can reserve or use vertices in Z to embed their children/grandchildren. ☐

In the following analysis we shall denote by $\tilde{Z}$ the set Z at the beginning of an iteration of the Algorithm 1. We also let $\hat{Z}$ denote the set Z just after line 1.22.

The algorithm does not abort at `find-critical-vertices`. Suppose that the algorithm aborts at line 4.6. This means that there is a set $\mathcal{C} \cup \mathcal{C}'$ such that each vertex $v \in \mathcal{C} \cup \mathcal{C}'$ sends at least $\alpha D_1/2$

edges into $(Z \cup D) \cap U$, which has size bounded by

$$|f_M(\mathcal{T}) \cap U| + |D| + (|\mathcal{C}| + |\mathcal{C}'|)\alpha 2^{-r_C} D_1 \leq n_0 + \varepsilon^3 N_0 + \frac{\varepsilon N_0}{4D_1}\alpha 2^{-r_C} D_1 < \frac{\alpha\varepsilon N_0}{4}.$$

Since in G' Property (‡).(ii) ensures that every subset of W having at most $\varepsilon N_0/(4D_1)$ elements expands by at least $(1-\varepsilon)D_1$, using Lemma 3, we must have $|(Z\cup D)\cap U| \geq (\alpha/2-2\varepsilon)D_1(\varepsilon N_0/D_1) > \alpha\varepsilon N_0/4$, a contradiction.

Now we show that the procedure does not abort at line 4.7. By the above argument, the set $\mathcal{C}'$ has cardinality at most $\varepsilon N_0/(4D_1)$. Moreover, we have that $(\hat{Z} \setminus \tilde{Z}) \cap U$ contains at most Δ_1 elements.

Observe that every call to `find-critical-vertices` is made with $Z = \hat{Z}$. Since vertices in $\mathcal{C}'$ were not critical, Invariant II implies that every $w \in \mathcal{C}'$ has degree at least $\alpha D_1/2 - \Delta_1$ on $\hat{Z}$.

Note that, although we consider the degree of vertices $w \in W \setminus \mathcal{C}$ on the set $Z \cup D \cup X$ in `find-critical-vertices` to classify a vertex as critical, the reserved neighborhood of new critical vertex may include recent dangerous vertices (those in $D \setminus Z$). The reason is that vertices which were just classified as dangerous still have reasonably large degree outside Z.

To prove that the desired disjoint family can be found, we invoke Lemma 3 to show that any subset of $\mathcal{C}'$ must have at least $\sim_{O(\varepsilon)} \alpha D_1/2$ neighbors outside of Z (in G) and then apply Lemma 12 to obtain a family covering the previous set X. (We may set $r_C \geq 2$.) □

The algorithm does not abort at line 1.19. If $p \notin \mathcal{C}$, then $|\Gamma_G(p) \setminus Z| \geq \alpha D_1/2$ because of Invariant II. In the remaining case, $p \in \mathcal{C} \setminus \mathcal{U}$, because of Invariant IV we have that $|S_p \setminus \mathcal{D}| \geq |S_p|/2 = \alpha 2^{-1-r_C} D_1$. In both cases, p has at least $\alpha 2^{-1-r_C} D_1$ neighbors u with $|\Gamma_G(u) \setminus Z| \geq \alpha D_0/2$.

If $\varepsilon N_1 \geq D_0 D_1$, apply Lemma 13 to $S \leftarrow \{p\}$, with $\alpha_{13} \leftarrow \alpha 2^{-1-r_C}$, $A_p \subseteq S_p \setminus \mathcal{D}$ with $|A_p| = \alpha_{13} D_1$ and $B_x \subseteq \Gamma_G(x) \setminus Z$ with $|B_x| = \alpha_{13} D_0$ for all $x \in A_p$. Refine the families obtained from Lemma 13 in such a way that the corresponding cardinalities match the degrees in the tree. This will produce the set S'_p and the disjoint family $\{Z_u\}_{u\in S'_p}$ of line 1.19.

Suppose that $\varepsilon N_1 < D_0 D_1$. We may require that every vertex in S_p has degree at most $(1+\varepsilon)D_0$ by possibly deleting at most εD_1 vertices from S_p (see Property (‡).(i)). Use Property (‡).(iv) applied to $S_p \setminus \mathcal{D} \subset \Gamma_G(p)$ to obtain disjoint sets $T_1, \ldots, T_r \subset S_p \setminus \mathcal{D}$. We need to find $S'_p = \{u_1, \ldots, u_\ell\} \subset S_p$ and a disjoint family $\{Z_i \subseteq \Gamma(u_i) \setminus Z\}_{i=1}^{\ell}$.

Given an arbitrary set X such that $|X| \leq \min\{n_1, \Delta_0\Delta_1\}$, we shall prove that the number of vertices $u \in T_i$ $(i = 1, \ldots, r)$ having $|\Gamma_G(u)\setminus(Z\cup X)| < \alpha D_0/4$ is at most $|T_i|/2$. Indeed, since $T_i \subseteq S_p \setminus \mathcal{D}$, we have $|\Gamma_G(u) \setminus Z| \geq \alpha D_0/2$ for all $u \in T_i$. Let $T'_i = \{u \in T_i : |\Gamma_G(u)\setminus(Z\cup X)| < \alpha D_0/4\}$. Note that every vertex $u \in T'_i$ must send

$$|\Gamma_G(u) \cap X| \geq |\Gamma_G(u) \setminus Z| - |\Gamma_G(u) \setminus (Z \cup X)| \geq \alpha D_0/4$$

edges into X.

Since T'_i is an LE set (by Property (‡).(iv)), we can apply Lemma 3 to show that $\min\{\Delta_0\Delta_1, n_1\} \geq |X| \geq \alpha D_0\, |T'_i|/8$. For C sufficiently large, it follows that

$$|T'_i| \leq \frac{8}{\alpha D_0} n_1 = \frac{8}{\alpha C D_0} N_1 \leq \frac{\varepsilon N_1}{8D_0}$$

and

$$|T'_i| \leq \frac{8}{\alpha D_0}\Delta_0\Delta_1 \leq \frac{8}{\alpha C^2} D_1 \leq \frac{\varepsilon D_1}{16},$$

thus $|T'_1| \leq \frac{1}{2}\min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\} = \frac{1}{2}|T_i|$.

We now construct S'_p and its corresponding disjoint family sequentially. Suppose that $u_1, \ldots, u_k$ have been selected from $\bigcup_{i=1}^r T_i$ together with a disjoint family $\{Z_i\}_{i=1}^k$. Set $X = \bigcup_{i=1}^k Z_i$ (initially $X = \emptyset$) and note that $|X| \le \Delta_0\Delta_1$ since $k \le \Delta_1$ and $|Z_i| \le \Delta_0$ for all i. It is also clear that $|X| \le n_1$ since to each vertex in X corresponds a vertex in $V_1(\mathcal{T})$.

By the above argument, at least half of the elements in $\bigcup_{i=1}^r T_i$ have large degree outside of $Z \setminus X$. Pick an arbitrary u_{k+1} (distinct from $u_1, \ldots, u_k$) having degree at least $\alpha D_0/4$. Set Z_{k+1} to be an arbitrary subset of $\Gamma_G(u_{k+1}) \setminus (Z \cup X)$ having the same number of elements as the number of children of u_k (which is at most $\Delta_0 < \alpha D_0/4$). Since $\ell \le \Delta_1 < \frac{1}{8}|S_p| < \frac{1}{2}\left|\bigcup_{i=1}^r T_i\right|$, it is always possible to extend the selection and the corresponding disjoint family. □

The algorithm does not abort at line 3.10. Suppose for the sake of a contradiction that the algorithm aborts because $\mathcal{U}$ grew larger than $\varepsilon N_1/(D_0D_1)$. Let us start with the case $\varepsilon N_1 \ge D_0D_1$. This means that we can find a set S of $\varepsilon N_1/(D_0D_1)$ elements together with a disjoint family of $\alpha 2^{-1-r_C} D_1$-sets $\{X_w \subseteq \Gamma_G(w)\}_{w\in S}$ where each $u \in X_w$ sends at least $\alpha D_0/2$ edges into $Z \cap W$. We may also enforce that every vertex in X_w has degree at most $(1+\varepsilon)D_0$ in G' by possibly deleting at most $\varepsilon D_1 < \alpha 2^{-2-r_C} D_1$ vertices from X_w (see Property (‡).(i)).

From Invariant I we know that

$$|Z \cap W| \le |f_M(\mathcal{T}) \cap W| + \frac{\varepsilon N_1}{D_0D_1}(\alpha^2 2^{-r_C-r_U} D_0D_1).$$

On the other hand, Property (‡).(iii) indicates that, if we take the set $T = \bigcup_{w\in S} X_w$, then $|T| = \alpha 2^{-2-r_C} D_1 |S| > \sqrt{\varepsilon} D_1 |S|$ and

$$d^*_{G'}(T) \ge (1 - O(\sqrt{\varepsilon}))D_0 |S| \cdot \alpha 2^{-2-r_C} D_1 = (1 - O(\sqrt{\varepsilon}))\varepsilon\alpha 2^{-2-r_C} N_1.$$

Since the degrees of the vertices of T (in G') are upper bounded by $(1+\varepsilon)D_0$, $|\Gamma_{G'}(T)| \le (1+\varepsilon)D_0 |T|$. Applying Lemma 3 over the graph $G[T, Z \cap W] \subset G'$, we obtain

$$\begin{aligned} |Z \cap W| &\ge |T|\frac{\alpha D_0}{2} + 2\{(1 - O(\sqrt{\varepsilon}))D_0 |T| - (1+\varepsilon)D_0 |T|\} \\ &= \left(\frac{\alpha}{2} - O(\sqrt{\varepsilon})\right) D_0 |T| \\ &\ge \alpha^2 2^{-3-r_C} N_1, \end{aligned}$$

a contradiction when $r_U \ge 4$ and C is sufficiently large.

For the case $\varepsilon N_1 < D_0D_1$, let w be the first ultra-critical vertex to be found. Apply Property (‡).(iv) to S_w. Let $T_1, \ldots, T_r \subset S_w$ be the disjoint sets obtained from the property. By assumption, there is a set $B \subseteq S_w$ with at least $|S_w|/2$ elements $u \in S_w$ with $\deg_G(u, Z \cap W) \ge \alpha D_0/2$. Since $\sum_{i=1}^r |T_i| \ge \frac{3}{4}|S_w|$, there exists some T_i with $|T_i \cap B| \ge |T_i|/4$. Because $T_i \cap B$ is an LE set, from Lemma 3 we obtain $|Z \cap W| \ge \alpha D_0 |T_i|/16$. This is a contradiction if we take C to be sufficiently large since $|T_i| = \min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\}$ and, by Invariant I, $|Z \cap W| \le n_1$. □

Claim 16. *For any $w \in W$, the number of vertices $u \in \Gamma_G(w)$ with $\deg_{G'}(u) \sim_\varepsilon D_0$ and $\deg_G(u, (\hat{Z} \setminus \tilde{Z}) \cap W) \ge \alpha D_0/4$ is at most $\sqrt{\varepsilon} D_1$.*

Proof of Claim 16. Given any $w \in W$, let us bound the number N_w of vertices $u \in \Gamma_G(w)$ such that $\deg_G(u, (\hat{Z} \setminus \tilde{Z}) \cap W) \ge \alpha D_0/4$. Since $(\hat{Z} \setminus \tilde{Z}) \cap W = \bigcup_{i=1}^l Z_i$ (see line 1.22), it is clear that $|(\hat{Z} \setminus \tilde{Z}) \cap W| \le \Delta_0\Delta_1$. If $N_w \ge \sqrt{\varepsilon} D_1$, by Property (‡).(iii) and Lemma 3, we should have $|(\hat{Z} \setminus \tilde{Z}) \cap W| \ge (\alpha/4 - O(\sqrt{\varepsilon}))\sqrt{\varepsilon} D_0D_1$, a contradiction. □

The algorithm does not abort at line 3.11. Note that $Z \cap W = \hat{Z} \cap W$ throughout the inner loop. Let $w \in \mathcal{U}'$. Since w was not ultra-critical before, by Invariant IV and equation (8), at least half of the elements $u \in S_w \in \mathcal{C}$ are such that $\deg_G(u, W \setminus \tilde{Z}) \ge \alpha D_0/2$. (It is possible that a vertex becomes critical and is promoted to ultra-critical during the execution of `restore-invariants`; the claim above is still true in that case since the reserved neighborhood for such a vertex would only contain vertices outside $\mathcal{D} \subseteq \tilde{Z}$.)

Since at most εD_1 vertices $u \in S_w$ fail to satisfy $\deg_{G'}(u) \sim_\varepsilon D_0$, by the Claim 16, less than $2\sqrt{\varepsilon} D_1$ neighbors of $w \in \mathcal{U}'$ may have more than $\alpha D_0/4$ edges going into $\hat{Z} \setminus \tilde{Z}$. Therefore, the number of $u \in S_w$ such that

$$\deg_G(u, W \setminus Z) = \deg_G(u, W \setminus \tilde{Z}) - \deg_G(u, \hat{Z} \setminus \tilde{Z}) \geq \alpha D_0/4 > \alpha 2^{-r_U} D_0$$

is greater than $|S_w|/4$. Since the family $\{Y_{w,u} \subseteq \Gamma(u) \setminus Z\}_{w \in \mathcal{U}', u \in S'_w}$ does not need to be disjoint, we are done. □

The algorithm does not abort at line 3.16. We shall apply Lemma 13 with $S \leftarrow R$, $\alpha_{13} \leftarrow \alpha 2^{-r_U}$, $A_w \subset S'_w$ with $|A_w| = \alpha_{13} D_1$ for all $w \in S$ and $B_x \subset Y_{w,x}$ with $|B_x| = \alpha_{13} D_0$ for all $w \in S$, $x \in A_w$. The families obtained through Lemma 13 are precisely the ones required at line 3.16. □

We have covered all invariants and all places where the algorithm could have aborted. Notice that the root of $\mathcal{T}$ is embedded and that, given any embedded vertex $v \in V_1(\mathcal{T})$, the children and grandchildren of v will be embedded at some point. This shows that the entire tree $\mathcal{T}$ can be embedded into G. ■

It is possible to apply Theorem 15 to every sufficiently dense subgraph of a graph satisfying Property (‡) by pre-processing the graph in a simple way.

Theorem 17. *Let $n_0, n_1, \Delta_0, \Delta_1$ be given. Suppose that G is a graph satisfying Property (‡) for some $\varepsilon > 0$, $N_0 = Cn_0$, $N_1 = Cn_1$ (with C sufficiently large) and $p = \max\{\Delta_0/n_1, \Delta_1/n_0\} < \varepsilon/8$.*

There exists an absolute constant $c > 0$ such that any subgraph $G' \subseteq G$ with $e(G') \geq c\sqrt{\varepsilon}\, e(G)$ contains every $(n_0, \Delta_0, n_1, \Delta_1)$-tree.

Proof. Let $D_0 = pN_1$ and $D_1 = pN_0$. Notice that, by assumption, $e(G) \sim_{\varepsilon^2} pN_0N_1 = D_0N_0 = D_1N_1$.

Let $W' \subseteq W(G)$ be the set described by Property (‡) and let $\alpha = 8\alpha_{15}$, where α_{15} is defined on Theorem 15. Suppose that $e(G') \geq 2\alpha\, e(G)$. By (‡).(v), we may assume that G' does not contain any edge incident to $W(G) \setminus W'$ by removing edges from G' while having $e(G') \geq \frac{3}{2}\alpha\, e(G)$ (the number of edges removed is upper bounded by $(1+\varepsilon^2)pN_0(2\varepsilon N_1) < 3\varepsilon\, e(G)$).

While there exists a vertex in $U(G')$ having degree less than $\alpha D_0/8$ or a vertex in $W(G') \subseteq W'$ having degree less than $\alpha D_1/8$, remove this vertex from G' together with all the edges incident to the removed vertex. The number of edges incident to removed vertices is upper bounded by $N_0(\alpha D_0/8) + N_1(\alpha D_1/8) \leq \frac{1}{2}\alpha\, e(G)$. Hence, the remaining G' is non-empty and we may apply Theorem 15. ■

From Theorem 17 we may prove Beck's conjecture.

Corollary 18 (Beck's Conjecture)**.** *The size-Ramsey of a tree $\mathcal{T}$ is $\Theta(\beta(\mathcal{T}))$.*

Proof. Given the constant c of Theorem 17, let $\varepsilon > 0$ be such that $c\sqrt{\varepsilon} < 1/2$. Without loss of generality, assume that $\varepsilon = 2^{-a}$ for some $a > 0$. Let $n_0, \Delta_0, n_1, \Delta_1$ be the parameters of $\mathcal{T}$. By possibly enlarging those values, we may assume that each of them is a power of 2. Since for every integer a there is an n such that $2^n \leq a < 2^{n+1}$, in the worst case, we may have to double each parameter. We may also assume that $n_0\Delta_0 = n_1\Delta_1$ by possibly increasing some Δ_i. These changes may only affect $n_0\Delta_0 + n_1\Delta_1$ by a multiplicative constant. The embedding algorithm is not affected since the parameters are only used as upper bounds on the cardinalities of the classes and their respective degrees.

Let $p = \Delta_1/n_0 = \Delta_0/n_1$. If $p \geq \varepsilon/8$ then we use the complete bipartite graph as our Ramsey graph (see Lemma 7).

If $p < \varepsilon/8$, we let $C = C(\varepsilon)$ be a sufficiently large constant and use Theorem 6 to obtain a graph G satisfying Property (‡) for ε, $N_0 = Cn_0$, $N_1 = Cn_1$ and p. By our choice of ε, from Theorem 17 we get that any subgraph $G' \subseteq G$ with at least $\frac{1}{2}e(G)$ edges contains $\mathcal{T}$.

Since in any two-coloring of the edges of G there will be one color containing at least half of its edges, the graph induced by the most frequent color contains $\mathcal{T}$. Moreover, we have

$$e(G) \leq 2pN_0N_1 = 2C^2pn_0n_1 = C^2(pn_1)n_0 + C^2(pn_0)n_1 = C^2(\Delta_0 n_0 + \Delta_1 n_1).$$

This shows that $\hat{r}(\mathcal{T}) = O(\beta(T))$. Together with the lower bound proved by Beck, the conjecture is proved. ∎

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