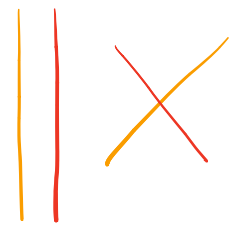
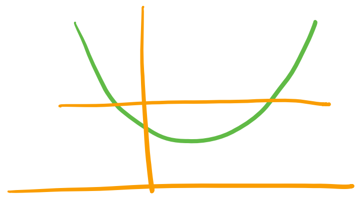
Projective space

Why projective space?

In $\mathbb{A}^2$, most pairs of lines intersect, but some don't:

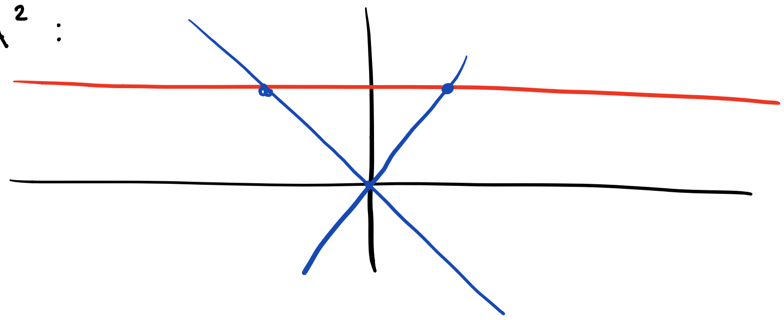


Most lines intersect a conic in 2 points, some in 1, some 0:



We want to add points to "compactify" affine space, so that curves can intersect "at infinity."

How to do this? We can identify each point $x \in \mathbb{A}^1$ w/ the point $(x,1)$ in $\mathbb{A}^2$:



Every such point uniquely determines a line in $\mathbb{A}^2$ that passes through $(0,0)$ and $(x,1)$.

Every line through the origin, other than $y=0$, corresponds to exactly one such point. $y=0$ corresponds to the "point at infinity". This set is called $\mathbb{P}^1$!

Def: Projective n-space over k, denoted $\mathbb{P}^n_k$, or just $\mathbb{P}^n$, is the set of all lines through the origin in $\mathbb{A}^{n+1}$.

Any point $(x_1, \dots, x_{n+1}) \neq (0, 0, \dots, 0)$ determines a unique such line, $\{(\lambda x_1, \dots, \lambda x_{n+1}) \mid \lambda \in k\}$.

$x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{n+1})$ determine the same line if and only if $\exists$ nonzero $\lambda \in k$ s.t. $y_i = \lambda x_i \;\; \forall i$. x and y are <u>equivalent</u> in this case.

<u>Alternate def</u>: $\mathbb{P}^n$ = the set of equivalence classes of points in $\mathbb{A}^{n+1} \setminus \{(0, \dots 0)\}$.

We write points in $\mathbb{P}^n$ as $[x_1 : \dots : x_{n+1}]$, called <u>homogeneous coordinates</u>.

<u>Note</u>: The <u>value</u> of x_i is not well-defined, but we can always say whether or not the i^{th} coordinate is 0.

If $x_j \neq 0$, the ratio x_i / x_j is well-defined.

<u>Ex</u>: $[1 : 0 : 2] = [2 : 0 : 4]$

<u>Covering $\mathbb{P}^n$ in $\mathbb{A}^n$s</u>

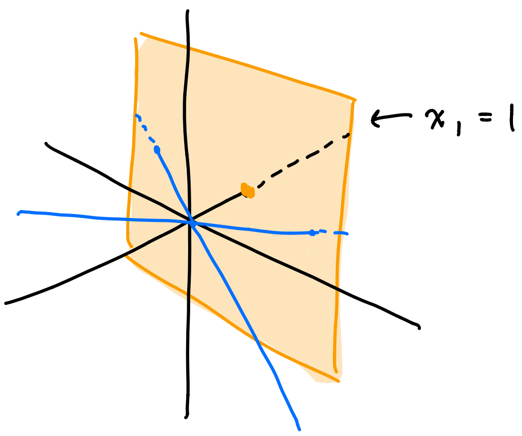
For each $i \in \{1, \dots, n+1\}$, define $U_i = \{[x_1 : \dots : x_{n+1}] \mid x_i \neq 0\} \subseteq \mathbb{P}^n$.

Then, by scaling, each $P \in U_i$ can be written uniquely as $P = [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_{n+1}]$.

Notice that since $x_i \neq 0$, there is no restriction on the other coordinates.

$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$ are called the nonhomogeneous coordinates for P w.r.t. U_i.

Ex: In $\mathbb{P}^2$, $U_1 = \{[1 : x_2 : x_3]\} = \{\text{lines in } \mathbb{A}^3 \text{ through } x_1 = 1 \text{ and the origin}\}$



This is in 1-to-1 correspondence w/ the plane $x_1 = 1$ $(\mathbb{A}^2)$

$\exists$ an injection $U_1 \hookrightarrow \mathbb{P}^2$

$$(x_2, x_3) \mapsto [1 : x_2 : x_3]$$

Let $P = [1 : 2 : 0]$

$$
\begin{array}{ccccc}
 & (2,0) & \longmapsto & [1:2:0] & \\
U_1 & \hookrightarrow & \mathbb{P}^2 & \| & \\
U_2 & \hookrightarrow & & [\tfrac{1}{2}:1:0] & \\
 & (\tfrac{1}{2},0) & \longmapsto & \| & \\
 & & & P &
\end{array}
$$

However $P \notin U_3$.

In general, if $P = [x_1 : \cdots : x_{n+1}] \in \mathbb{P}^n$, $\exists\, i$ s.t. $x_i \neq 0$, so $P \in U_i$.

$$\Rightarrow \mathbb{P}^n = \bigcup_{i=1}^{n+1} U_i$$

On the other hand, define $H_\infty = \mathbb{P}^n \setminus U_{n+1} = \{[x_1 : \cdots : x_{n+1}] \mid x_{n+1} = 0\}$, called the <u>hyperplane at infinity</u>.

Now we don't have a fixed representative for each point in H_∞.

Have 1-to-1 correspondence $H_\infty \leftrightarrow \mathbb{P}^{n-1}$

$[x_1 : \dots : x_n : 0] \leftrightarrow [x_1 : \dots : x_n]$

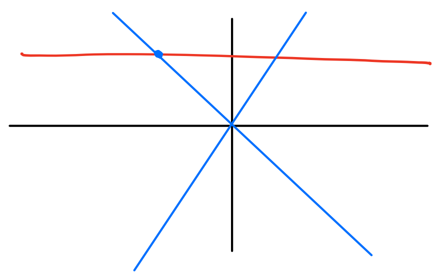
$$\text{So } \mathbb{P}^n = \underset{\mathbb{A}^n}{\underset{\cong}{U_{n+1}}} \cup \underset{\mathbb{P}^{n-1}}{\underset{\cong}{H_\infty}}$$

U_{n+1}: lines through origin and hyperplane $x_{n+1} = 1$ in $\mathbb{A}^{n+1}$

H_∞: lines through origin in hyperplane $x_{n+1} = 0$

Ex: 1.) $\mathbb{P}^0 = \{\text{lines through origin in } \mathbb{A}^1\} = \{[\alpha] \mid \alpha \neq 0\} = $ a point.

2.) $\mathbb{P}^1 = \mathbb{A}^1 \cup \{pt\}$ = projective line



3.) Consider the line L defined by $y = mx + b$ in $\mathbb{A}^2$

What equation(s) defines this line in $\mathbb{P}^2$?

Identify $\mathbb{A}^2$ w/ $U_3 \subseteq \mathbb{P}^2$

$$L = \{(x,y) \mid y = mx + b\} = \{[x:y:1] \mid y = mx+b\} = \{[x:y:z] \mid \underset{z \neq 0}{y = mx + bz}\}$$

Problem: $\alpha y \neq m\alpha x + b$

but $\alpha y = m\alpha x + b\alpha$

Let $L' = \{[x:y:z] \in \mathbb{P}^2 \mid y = mx + bz\}$ Then $L' \cap U_3 = L$ and $L' \cap H_\infty = \{[1:m:0]\}$

i.e. all lines w/ the same slope meet at the same point at infinity.

4.) Consider the curve $y^2 = x^2 + 1$ in $\mathbb{A}^2$. The corresponding set is given in $\mathbb{P}^3$ by $y^2 = x^2 + z^2$ (if $z = 1$, we're in $(U_3 = \mathbb{A}^2)$)

This intersects H_∞ when $y^2 = x^2$ i.e. $x = y$ or $x = -y$

i.e. $[1:1:0]$ and $[1:-1:0]$