Projective plane curves

Def: A <u>projective plane curve</u> is an equivalence class of forms in $k[x,y,z]$ s.t. $F \sim G \iff F = \lambda G$ for some $\lambda \in k$ nonzero.

All notation, definitions, conventions carry over from affine curves.

Remark: Note that if F is a projective curve and $P = [x:y:1]$, then $\mathcal{O}_P(F) \cong \mathcal{O}_P(f)$, where $f = F(x,y,1) \in k[x,y]$.

We define the multiplicity of F at P in this case to be $m_P(F) := m_P(f)$.

Recall: If $f = \overset{\neq 0}{f_m} + \dots f_{m+r}$ is an affine curve, $P = (0,0)$, then the multiplicity of f at P is m.

Claim: F a plane curve, then P is a multiple point of F
$\iff F(P) = F_x(P) = F_y(P) = F_z(P) = 0$.

Pf: WLOG, assume $P \in U_3$ so $P = [a:b:1]$.

Restricting to U_3, P is a multiple point $\iff F(P) = F_x(P) = F_y(P) = 0$.

If $P \neq [0:0:1]$, then $P \in U_i$ for some $i = 1$ or 2, and we get $F_z(P) = 0$.

If $P = [0:0:1]$, then $F = xf + yg + \lambda z^d \Rightarrow \lambda = 0 \Rightarrow F_z = xf_z + yg_z \Rightarrow F_z(P) = 0$.

$\square$

Def: Let F be a plane curve, P a point in F, $P \in U_i$. Let L be a line through P. Let f and ℓ be the dehomog. of F and L w.r.t. x_i. Then L is <u>tangent</u> to F at $P \iff \ell$ is tangent to f at P.

Ex: $F = xy^4 + yz^4 + xz^4$.

$F_x = y^4 + z^4 = 0$

$F_y = 4xy^3 + z^4 = 0$

$F_z = 4yz^3 + 4xz^3 = 4z^3(y+x) = 0 \Rightarrow z = 0$ or $y = -x$

If $z = 0$, then $y = 0$, and $F(1,0,0) = 0$, so $[1:0:0]$ is a mult. point.

If $x = -y$, then $-y^4 = z^4 = 4y^4 \Rightarrow y = x = 0 \Rightarrow z = 0$, which doesn't work, so $[1:0:0]$ is the only multiple point.

$[1:0:0] \in U_1$, and dehomogenizing gives $f = y^4 + z^4 + yz^4$, which has multiplicity 4 at $[1:0:0]$, and the tangent lines are the 4 factors of $y^4 + z^4$.

Def: Let F, G be projective plane curves, $P \in \mathbb{P}^2$. Choose i s.t. $P \in U_i$. The <u>intersection of F and G</u> at P is

$$I_p(F, G) := I_p(f, g),$$

where f and g are F and G dehomogenized w/ respect to U_i.

Remark: $I_p(F,G)$ satisfies all the properties of the affine intersection #, except:

In 3.) T should be a projective change of coordinates.

In 7.), which says $I_p(F,G) = I_p(F, G+AF)$, A should be a form w/ $\deg A = \deg G - \deg F$.