Regular maps

Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be algebraic sets.

Def: A function $\varphi : V \to W$ is a <u>regular map</u> (or <u>polynomial map</u> or <u>morphism</u>) if there are $T_1, \dots, T_m \in k[x_1, \dots, x_n]$ s.t. for all $a := (a_1, \dots, a_n) \in V$

$$\varphi(a) = (T_1(a), T_2(a), \dots, T_m(a)).$$

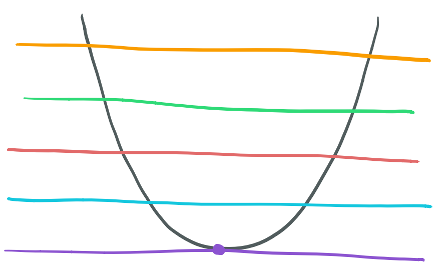
Ex: Consider $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ defined $t \mapsto (t, t^2)$. This has image $V = V(y - x^2)$.

Define $\psi : V \to \mathbb{A}^1$ by

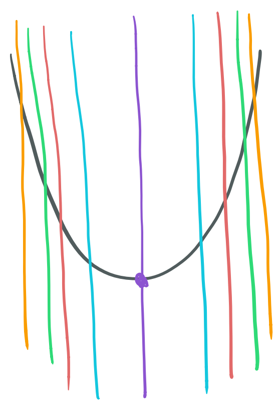
$(x, y) \mapsto y$ <u>or</u>

$(x, y) \mapsto x^2$ (both give the same map)

Note that both of these descriptions of ψ extend to $\mathbb{A}^2$, but not in the same way.



The colored lines represent the fibers of the two extensions to $\mathbb{A}^2$



Note: A regular function f on V determines a regular map $V \to \mathbb{A}^1$.

Q: How are morphisms of varieties related to the corresponding coordinate rings?

If we have $\varphi: V \to W$ a regular map, we get a k-algebra homomorphism:

$$\varphi^*: \Gamma(W) \to \Gamma(V)$$

If $g \in \Gamma(W)$, then
$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[rr, bend right, "g \circ \varphi"'] & W \arrow[r, "g"] & k \end{tikzcd}$$

We define $\varphi^*(g) := g \circ \varphi$

φ^* is called the pullback of φ (Fulton denotes this $\tilde{\varphi}$)

Ex: Consider the morphism
$$\mathbb{A}^3 \to \mathbb{A}^2$$
$$(x,y,z) \mapsto (x^2y, x-z)$$

The pullback is defined
$$k[u,v] \to k[x,y,z]$$
$$u \mapsto x^2y$$
$$v \mapsto x-z$$

Remark: If $\varphi: V \to W$, $\psi: W \to X$ are regular, then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*:$$

$$V \xrightarrow{\varphi} W \xrightarrow{\psi} X \xrightarrow{f} k$$

If $f \in \Gamma(X)$, then $(\psi \circ \varphi)^*(f) = f \circ (\psi \circ \varphi)$ and

$$\varphi^* \circ \psi^*(f) = \varphi^*(f \circ \psi) = (f \circ \psi) \circ \varphi$$

Also, note that if $id: X \to X$ is the identity, then $id^*(f) = f \circ id = f$.

Together, these facts tell us that Γ is a <u>contravariant functor</u> from the category of algebraic sets to reduced (i.e. $\sqrt{(0)} = (0)$) finitely-generated k-algebras.

Ex: If $X \subseteq \mathbb{A}^n$, and $f: X \to \mathbb{A}^n$ the corresponding inclusion, then $f^*(x_i) = \overline{x}_i \in \Gamma(X)$, so the map is just the natural quotient.

Exer: If $V \subseteq W$, then the corresponding pullback $\Gamma(W) \to \Gamma(V)$ is the obvious quotient.

Prop: Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be algebraic sets. There is a one-to-one correspondence between regular maps from V to W and k-algebra homomorphisms $\Gamma(W) \to \Gamma(V)$.

Pf: Let $\alpha: \Gamma(W) \to \Gamma(V)$ be a homomorphism.

$\Gamma(W) = \dfrac{k[y_1, \ldots, y_m]}{I(W)}$, $\Gamma(V) = \dfrac{k[x_1, \ldots, x_n]}{I(V)}$

We want to construct a map $\varphi: V \to W$.

i.e. $\varphi = (\varphi_1, \ldots, \varphi_m)$, where $\varphi_i : V \to k$, so $\varphi_i \in \Gamma(V)$

Define $\varphi_i = \alpha(y_i)$.

We need to first check that $\Psi(V) \subseteq W$:

Let $p \in V$. WTS $\Psi(p) \in W$.

Take $g \in I(W)$.

$$g(\Psi(p)) = g(\alpha(y_1)(p), \dots, \alpha(y_m)(p)) = \underset{\text{since } g \in I(W)}{\alpha(g)(p) = 0}$$

Thus, $\Psi(p) \in V(I(W)) = W$.

Now we need to show $\Psi^* = \alpha$.

If $f \in \Gamma(W)$, $p \in V$, WTS $\Psi^*(f)(p) = \alpha(f)(p)$

$\Psi^*(f)(p) = f(\Psi(p)) = \alpha(f)(p)$, so $\Psi^* = \alpha$.

So we showed $\{V \to W\} \xrightarrow{*} \{\Gamma(W) \to \Gamma(V)\}$ is surjective.

Check (on HW): If $\Psi, \varphi : V \to W$ s.t. $\Psi^* = \varphi^*$, then $\Psi = \varphi$.
(Hint: If $p \in V$ s.t. $\Psi(p) \neq \varphi(p)$, find $f \in \Gamma(W)$ that vanishes at one but not the other.) $\square$

Ex: Consider the projection $\varphi: \mathbb{A}^n \to \mathbb{A}^r$ $n \geq r$ defined

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_r)$$

This is clearly a regular map, and its pull back is the inclusion

$$k[y_1, \dots, y_r] \to k[x_1, \dots x_n]$$
$$y_i \longmapsto x_i$$

Ex: Consider the k-algebra map

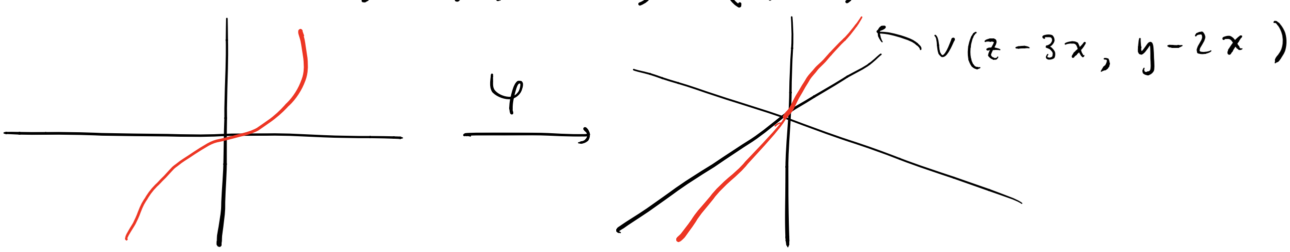
$$\alpha : k[x,y,z] \to k[u,v]/(u-v^3)$$
$$x \mapsto u$$
$$y \mapsto 2u$$
$$z \mapsto 3u$$

(Note that this is well-defined!)

Let $V = V(u-v^3)$.

What is φ s.t. $\varphi^* = \alpha$?

$$\varphi(u,v) = (\alpha(x), \alpha(y), \alpha(z)) = (u, 2u, 3u)$$



Def: A regular map $\varphi : V \to W$ is an <u>isomorphism</u> if $\exists$ an inverse $\psi : W \to V$ that's also regular.

(Note: It's not enough for φ to be bijective — see hw.)

Cor (of prop) $\varphi : V \to W$ is an isomorphism.

$$\iff \varphi^* : \Gamma(W) \to \Gamma(V) \text{ is an isomorphism.}$$

Pf: Let $\psi : W \to V$. Then $\varphi \circ \psi = id \iff \psi^* \circ \varphi^* = (\varphi \circ \psi)^* = id$

So φ is an isom. $\iff \varphi^*$ is. $\square$

Ex: Consider $\varphi : \mathbb{A}^1 \to V(y - x^2) = V$
$t \longmapsto (t, t^2)$

Then $\psi : V \longrightarrow \mathbb{A}^1$

$(x, y) \longmapsto x$ is algebraic.

$\Rightarrow \psi = \varphi^{-1}$.

Alternatively, can check that $\varphi^* : k[x,y]/(y - x^2) \to k[t]$ is an
$x \longmapsto t$
$y \longmapsto t^2$

isomorphism.

Ex: $T = (T_1, \ldots, T_n)$ regular map $\mathbb{A}^n \to \mathbb{A}^n$ where each T_i is deg 1 and T is a bijection. T is called an <u>affine change of coordinates</u>.

We can decompose $T = \underset{\text{translation}}{T''} \circ \underset{k\text{-linear}}{T'}$ where T' is invertible (as a linear map)

Both have regular inverses, so T is an isomorphism.

Lemma: Let $\varphi : V \to W$ be regular, $X \subseteq W$ algebraic.

a.) $\varphi^{-1}(X)$ is algebraic.

b.) If $X \subseteq \varphi(V)$ and $\varphi^{-1}(X)$ irreducible, then X is irreducible.

Pf: a.) $X = V(f_1, \ldots, f_r) = V(f_1) \cap \ldots \cap V(f_r)$, so

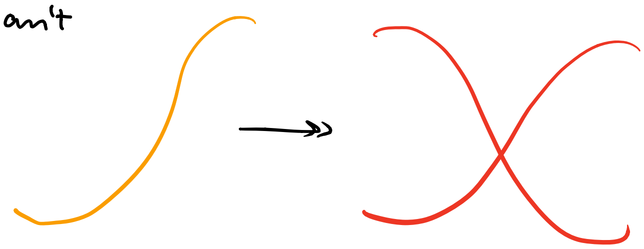
$\varphi^{-1}(X) = \bigcap \varphi^{-1}(V(f_i))$

$$\varphi^{-1}(V(f_i)) = \{a \in V \mid f_i(\varphi(a)) = 0\} = \{a \in V \mid \varphi^*(f_i)(a) = 0\}$$
$$= V(\varphi^*(f_i))$$

b.) Suppose $X = A \cup B$, A, B alg. sets.

$$\varphi^{-1}(X) = \varphi^{-1}(A) \cup \varphi^{-1}(B). \text{ WLOG } \varphi^{-1}(A) = \varphi^{-1}(X) \Rightarrow A = X. \ \square$$

i.e. we can't have



Note: The image of a reducible set can be irreducible $(V(xy) \twoheadrightarrow V(x))$

Injectivity + Surjectivity

Prop: V and W algebraic, $\varphi: V \to W$ regular and surjective $\Rightarrow \varphi^*$ is injective.

Pf: Let $f \in \Gamma(W)$ s.t. $\varphi^*(f) = 0$. Then

$$\begin{tikzcd} V \arrow[r, two heads] \arrow[rr, bend right, "0"'] & W \arrow[r, "f"] & k \end{tikzcd} \Rightarrow f = 0. \ \square$$

Note: The converse doesn't hold!

Ex: Consider $V = V(xy-1) \subseteq \mathbb{A}^2$ and define

$$\varphi: V \to \mathbb{A}^1 \text{ by}$$
$$(x,y) \mapsto x$$

Then $\varphi^* : k[t] \to k[x,y]/(xy-1)$,

$t \longmapsto x$

which is injective, but φ isn't surjective. $(0 \notin \varphi(V))$

In fact, the image is not an algebraic set in $\mathbb{A}^1$!
(This is not something that happens in projective space!)

We get an if and only if statement if we consider the closure of the image.

Def: $\varphi : V \to W$ regular. φ is <u>dominant</u> if $\overline{\varphi(V)} = W$.
(Equivalently, $I(\varphi(V)) = I(W)$)

Prop: $\varphi : V \to W$ regular. φ^* is injective $\Leftrightarrow$ φ is dominant.

Pf: Assume φ^* injective. Let $f \in I(\varphi(V))$.

Then $\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi^*(f)"'] & W \arrow[r, "f"] & k \end{tikzcd}$ $\Rightarrow$ if $\varphi^*(\bar{f}) = 0$, then $f \in I(W)$.

Now assume φ is dominant. Suppose $\bar{f} \in \Gamma(W)$ and $\varphi^*(\bar{f}) = 0$.

$\Rightarrow$ $f \in I(\varphi(V)) = I(W)$ $\Rightarrow$ $\bar{f} = 0$.

Prop: $\varphi^* : \Gamma(W) \to \Gamma(V)$ is surjective $\Leftrightarrow$ φ is an isomorphism onto its image.

Pf: $\Leftarrow$: Let $\varphi(V) = W'$.

$$\begin{tikzcd} V \arrow[d, "\cong"'] \arrow[dr, "\varphi"] & \\ W' \arrow[r, phantom, "\subseteq"] & W \end{tikzcd} \qquad \begin{tikzcd} \Gamma(V) & \\ \Gamma(W') \arrow[u, "\cong"] & \Gamma(W) \arrow[l, two heads, "\text{quotient}"] \arrow[ul] \end{tikzcd}$$

Thus, this is surjective.

$\Rightarrow$: Let $W' = \overline{\varphi(V)}$. Then $V \to W'$ is dominant.

$$\begin{tikzcd} \Gamma(V) & \\ \Gamma(W') \arrow[u, hook] & \Gamma(W) \arrow[l, two heads] \arrow[ul, two heads] \end{tikzcd}$$

Thus, $\Gamma(W') \to \Gamma(V)$ is surjective and thus an isomorphism, so $V \to W'$ is an isomorphism. $\square$