Math 221 — Midterm Exam 1

October 1, 2021

Name: Solutions

Section: ________________

1. **Do not open this exam until you are told to do so.**
2. This exam has 6 pages including this cover. There are 5 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. You may use no aids (e.g., calculators or notecards) on this exam.

Problem	Points	Score
1	8	
2	10	
3	6	
4	5	
5	6	
Total	35	

1. [8 points] Let A be an invertible 4×4 matrix with inverse

$$A^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

a. [2 points] Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ Solve the system of equations $A\mathbf{x} = \mathbf{b}$.

$$\vec{x} = A^{-1}b = \begin{bmatrix} -7 \\ 1 \\ 3 \end{bmatrix}$$

b. [4 points] Write the matrix A as the product of elementary matrices. You do not need to compute A.

$$A^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \underbrace{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}_{= E_1 A^{-1}} \xrightarrow{2R_3} \underbrace{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= E_2 E_1 A^{-1}} \xrightarrow{R_1 + 3R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= I}$$

$$I = \underbrace{E_3 E_2 E_1}_{A} A^{-1}$$

So $A = E_3 E_2 E_1$, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c. [2 points] Find a matrix B such that $AB = \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$.

$$B = A^{-1}(AB) = \begin{bmatrix} -7 & 2 \\ 3/2 & 0 \\ -1 & 0 \end{bmatrix}$$

2. [10 points] For each part **a.** - **d.**, give **an example** of a matrix A satisfying the given property. You do not need to justify your answer.

a. [2 points] The system of equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has infinitely many solutions and A is not the zero matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. [2 points] The system of equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solutions and A is not the zero matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c. [2 points] A is the product of two elementary matrices but is not an elementary matrix itself. (You don't need to give the elementary matrices, just A.)

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

d. [2 points] A is it's own inverse (but is not the identity matrix).

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

e. [2 points] A is a 4×4 matrix, is not the zero matrix, and is not invertible.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. [6 points]

a. [5 points] Find all solutions to the system of equations

$$\begin{aligned} x + y + 6z &= 3 \\ x + 2y + 7z &= 4 \\ -2x - 2y - 12z &= -6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 1 & 2 & 7 & 4 \\ -2 & -2 & -12 & -6 \end{array}\right] \xrightarrow{R_3+2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 1 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 6 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 5z = 2$

$y + z = 1$

Set $z = t$

Solutions:

$x = 2 - 5t$

$y = 1 - t$

$z = t$

For all real values of t

b. [1 point] Give any specific solution to the system of equations in **b.**. (There may be more than one correct answer.)

setting $t = 0$, one solution is

$x = 2, y = 1, z = 0$

4. [5 points] For **a.** - **e.** circle **TRUE** or **FALSE**. You don't need to justify your answer.

a. [1 point] If A is a 2×4 matrix, and B is a 4×3 matrix, then the composition of the corresponding transformations $T_B \circ T_A$ is defined.

TRUE **FALSE**

(BA not defined)

b. [1 point] If A is a 3×4 matrix of rank 3, and $\mathbf{b}$ is a vector in $\mathbb{R}^3$, then the system of equations $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.

(leading 1 in each row, one nonleading variable)

TRUE **FALSE**

c. [1 point] If $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution to a system of equations, then the system of equations is homogeneous.

TRUE **FALSE**

d. [1 point] If $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$, there is an elementary matrix E such that $B = EA$.

$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

TRUE **FALSE**

e. [1 point] If A is a 3×4 matrix, $\mathbf{b}$ is a vector in $\mathbb{R}^3$, and the solution to the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly 2 parameters, then A has rank 1.

(2 leading variables, 2 nonleading variables)

TRUE **FALSE**

5. [6 points] Let $\mathbf{a} = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}1\\2\\0\end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix}2\\5\\m\end{bmatrix}$.

a. [4 points] Find a value of m so that $\mathbf{c}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

$$\left[\begin{array}{cc|c}1 & 1 & 2\\1 & 2 & 5\\2 & 0 & m\end{array}\right] \xrightarrow{R_3-2R_1} \left[\begin{array}{cc|c}1 & 1 & 2\\1 & 2 & 5\\0 & -2 & m-4\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c}1 & 1 & 2\\0 & 1 & 3\\0 & -2 & m-4\end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{cc|c}1 & 1 & 2\\0 & 1 & 3\\0 & 0 & m+2\end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 3\\0 & 0 & m+2\end{array}\right]$$

consistent if $m+2=0$

i.e. if $\boxed{m=-2}$

b. [2 points] Using the value of m you found in part **a.**, express $\mathbf{c}$ as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

$$-1\vec{a} + 3\vec{b} = \vec{c}$$