Math 221 — Midterm Exam 2

November 5, 2021

Name: Solutions

Section: ______

1. **Do not open this exam until you are told to do so.**
2. This exam has 7 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Unless explicitly stated otherwise, you should show your work for each problem.
4. You may use no aids (e.g., calculators or notecards) on this exam.

Problem	Points	Score
1	8	
2	3	
3	8	
4	5	
5	5	
6	6	
Total	35	

1. [8 points] Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$.

a. [2 points] Find the eigenvalues of A.

$$c_A(x) = \det(xI - A)$$

$$= \det \begin{bmatrix} x-1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 1 & x \end{bmatrix}$$

$$= (x-1)(x-1)x$$

$\Rightarrow$ eigenvalues are $\lambda_1 = 1, \lambda_2 = 0$

b. [4 points] Find all eigenvectors corresponding to each eigenvalue you found in **a.**

$\underline{\lambda_1 = 1}$:

$$(\lambda_1 I - A)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow y = -z$$

setting $x = s$, $z = t$, the solutions are

$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$ so 1-eigenvectors are $s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,

s, t not both 0.

$\underline{\lambda_2 = 0}$:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow x = z, y = 0$$

setting $z = t$, the 0-eigenvectors are

$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$, or $t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $t \neq 0$.

Problem **1.** continues on this page.

c. [2 points] Is A diagonalizable? If yes, find an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. If no, explain why.

Yes.

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. [3 points] Consider the two lines in $\mathbb{R}^3$ with vector equations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Do the two lines intersect? If so, at which point(s)? If not, how do you know?

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is on both lines, then

$$\begin{aligned} -s &= 1 + t \\ 3s &= 9 \\ 2s &= -1 - 2t \end{aligned}$$

for some s, t. This gives the following augmented matrix:

$$\left[\begin{array}{cc|c} -1 & -1 & 1 \\ 3 & 0 & 9 \\ 2 & 2 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 1 \\ 3 & 0 & 9 \\ 0 & 0 & 1 \end{array}\right],$$

which is inconsistent.

Thus, no such s, t exist, so the lines do not intersect.

3. [8 points] Short answer. For each part of this problem, you do not need to justify your answer.

a. [2 points] Let A be a 2×2 matrix. If $\begin{bmatrix}1\\0\end{bmatrix}$ is a 2-eigenvector of A and $\begin{bmatrix}0\\1\end{bmatrix}$ is a (-3)-eigenvector of A, what is A?

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

b. [2 points] If B is a 4×4 matrix, and $\det(2B) = -8$, what is $\det(B)$?

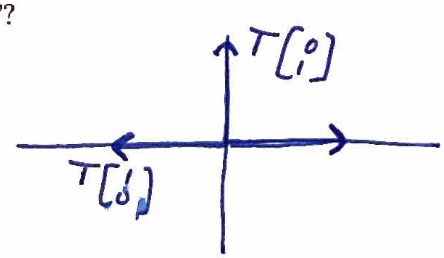
$$-8 = \det(2B) = 2^4 \det B = 16 \det B$$

$$\Rightarrow \det B = \frac{-1}{2}$$

c. [2 points] $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection over the x-axis followed by counterclockwise rotation by π. What is the matrix corresponding to T?

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$



$$\Rightarrow \text{matrix is } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

d. [2 points] Find a vector equation for the line through the points $(1, 1, -1)$ and $(0, 3, 4)$.

$$\vec{d} = \begin{bmatrix} 1-0 \\ 1-3 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

4. [5 points] For **a.** - **e.** circle **TRUE** or **FALSE**. You don't need to justify your answer.

a. [1 point] If A is a diagonalizable matrix, then it must be invertible.

TRUE **FALSE**

($\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable)

b. [1 point] If A and B are $n \times n$ matrices, and AB is not invertible, then at least one of A and B is not invertible.

TRUE **FALSE**

($\det(AB) = 0 \iff \det A \det B = 0$)

c. [1 point] Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to counterclockwise rotation by $\theta = \pi/2021$. The corresponding matrix A is diagonalizable.

TRUE **FALSE**

(No A-invariant lines)

d. [1 point] The function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x+1 \end{bmatrix}$ is a linear transformation.

TRUE **FALSE**

($T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

e. [1 point] Let A be an $n \times n$ matrix, and B the matrix obtained from A by switching the first two rows. If $\det A = \det B$, then A must be invertible.

TRUE **FALSE**

$\det A = -\det B = \det B$

$\Rightarrow \det B = 0 = \det A$

5. [5 points] Let $A = \begin{bmatrix} 3 & -1 & -c & 2 \\ 4 & 1 & 0 & 0 \\ c & c & -2 & c \\ 2 & 1 & 0 & 0 \end{bmatrix}$, where c is some real number.

a. [3 points] Compute $\det A$.

$$\det A = \det \begin{bmatrix} 3 & -1 & -c & 2 \\ 2 & 0 & 0 & 0 \\ c & c & -2 & c \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

$$= -2 \begin{vmatrix} -1 & -c & 2 \\ c & -2 & c \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -2 \left(1 \begin{vmatrix} -c & 2 \\ -2 & c \end{vmatrix} \right)$$

$$= -2(-c^2 + 4)$$

$$= 2c^2 - 8$$

b. [2 points] For which values of c is A invertible?

A is invertible $\Leftrightarrow$ $2c^2 - 8 \neq 0$

$\Leftrightarrow$ $c^2 \neq 4$, i.e. $c \neq \pm 2$

6. [6 points] Suppose T is a linear transformation such that $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$ and $T\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$.

a. [3 points] Find the matrix A corresponding to T.

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}0\\-1\end{bmatrix}$$

So $T\begin{bmatrix}1\\0\end{bmatrix} = T\begin{bmatrix}1\\1\end{bmatrix} + T\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$

$$T\begin{bmatrix}0\\1\end{bmatrix} = T\left(-\begin{bmatrix}0\\-1\end{bmatrix}\right) = -T\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$$

So $A = \begin{bmatrix}2 & -1\\0 & -1\end{bmatrix}$

b. [3 points] Is the matrix you found in **a.** diagonalizable? Justify your answer.

$$C_A(x) = \det\begin{bmatrix}x-2 & 1\\0 & x+1\end{bmatrix} = (x-2)(x+1).$$

So A has two distinct eigenvalues. Since A is 2×2, this implies it is diagonalizable.