Math 421 Problem Set
September 29, 2022

1. Let G be a finite group.

   (a) If $|G| = 3$, show that G is cyclic.

   (b) If $|G| = 4$, show that G is either cyclic, or isomorphic to $Z_2 \times Z_2$, the product of two cyclic groups of order 2.

   (c) Show that if G has prime order, it must be cyclic.

   (d) Show that G has even order if and only if it has an element of order 2. [Hint: Consider the set $\{x \in G \mid x \neq x^{-1}\}$. How many elements does it have? What is its complement?]

2. Let n be a positive integer greater than 1. For each $a \in \mathbb{Z}$, define the homomorphism $\sigma_a : Z_n \to Z_n$ by $\sigma_a(x) = x^a$.

   (a) Show that σ_a is an automorphism (recall definition from Tuesday's homework) if and only if a and n are relatively prime.

   (b) Show that $\sigma_a = \sigma_b$ if and only if $a \equiv b \pmod{n}$.

   (c) Prove that every automorphism of Z_n is equal to σ_a for some integer a.

   (d) Prove that $\sigma_a \circ \sigma_b = \sigma_{ab}$.

   (e) Deduce that the map $\bar{a} \to \sigma_a$ is an isomorphism of $(\mathbb{Z}/n\mathbb{Z})^\times$ onto the automorphism group of Z_n.