Math 421 Problem Set
October 4, 2022

1. A group G is ***finitely generated*** if there is a finite set A such that $G = \langle A \rangle$.

   (a) Show that every finitely generated subgroup of $\mathbb{Q}$ (under addition) is cyclic.

   (b) Show that $\mathbb{Q}$ is not finitely generated.

   (c) Find a proper subgroup of $\mathbb{Q}$ (i.e. $H \leq \mathbb{Q}$ such that $H \neq \mathbb{Q}$) that is not cyclic.

2. A subgroup M of a group G is a ***maximal subgroup*** if $M \neq G$ and the only subgroups of G that contain M are M and G.

   (a) Show that if H is a proper subgroup of a finite group G, then there is a maximal subgroup of G containing H.[1]

   (b) Show that the subgroup of all rotations in a dihedral group is a maximal subgroup.

   (c) Show that if $G = \langle x \rangle$ is a cyclic group of order n, then a subgroup H is maximal if and only if $H = \langle x^p \rangle$ for some prime p dividing n.

[1] This is actually true in any finitely generated group – the proof uses Zorn's Lemma.