Math 421 Problem Set
October 27, 2022

1. Let $G = Z_4 \times Z_4$, which has the following presentation:

$$G = \langle x, y \mid x^4 = y^4 = 1, xy = yx \rangle.^1$$

Let $\bar{G} = G/\langle x^2y^2 \rangle$ (note that every subgroup of an abelian group is normal). For $g \in G$, denote the coset $g\langle x^2y^2 \rangle$ by $\bar{g}$.

   (a) Show by Lagrange's Theorem that $|\bar{G}| = 8$.

   (b) Write each element of $\bar{G}$ in the form $\bar{x}^a\bar{y}^b$ for some integers a and b.

   (c) Find the order of each of the elements of $\bar{G}$.

   (d) Show that $\bar{G} \cong Z_4 \times Z_2$.

2. Let G be a group. We showed in class that $Z(G) \trianglelefteq G$.

   (a) Show that if $G/Z(G)$ is cyclic, then G is abelian. *[Hint: Let $xZ(G)$ be a generator. Then every element of G can be written in the form x^az for some $a \in \mathbb{Z}$ and $z \in Z(G)$.]*

   (b) Show that if $|G| = pq$ for some primes p and q (not necessarily distinct), then either G is abelian or $Z(G) = 1$.

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[1]Note that instead of writing the generators as $(x, 1)$ and $(1, y)$, we are just writing x and y to make notation easier.