Math 421 Problem Set
November 3, 2022

1. Let H and K be subgroups of finite index in the group G, with $|G : H| = m$ and $|G : K| = n$.

   (a) Let k be the least common multiple of m and n. Show that

   $$k \leq |G : H \cap K| \leq mn.$$

   (b) Deduce that if m and n are relatively prime, then $|G : H \cap K| = mn$.

   (c) If $H \leq K \leq G$, show that $|G : H| = |G : K||K : H|$.

2. Let M and N be normal subgroups of G such that $G = MN$.

   (a) Show that for any elements $m \in M$ and $n \in N$, there are elements $m' \in M$ and $n' \in N$ such that $mn' = nm'$.

   (b) Prove that

   $$G/(M \cap N) \cong (G/M) \times (G/N).$$

   [Hint: Come up with a map $G \to (G/M) \times (G/N)$ and show it's surjective. What is its kernel?]