Math 421 Problem Set
August 30, 2022

1. Let $n > 1$. Recall that $\mathbb{Z}/n\mathbb{Z}$ is the set of residue classes mod n.

   (a) Check that multiplication in $\mathbb{Z}/n\mathbb{Z}$ is well-defined; that is, if $\bar{a_1} = \bar{b_1}$ and $\bar{a_2} = \bar{b_2}$, then $\overline{a_1 \cdot a_2} = \overline{b_1 \cdot b_2}$.

   (b) Show that $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication.

   (c) We define
   $$(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid \text{there is } \bar{c} \in \mathbb{Z}/n\mathbb{Z} \text{ such that } \bar{a} \cdot \bar{c} = \bar{1}\}.$$
   Check that $(\mathbb{Z}/n\mathbb{Z})^\times$ is a group under multiplication. (You can assume that the standard multiplication on $\mathbb{Z}$ is associative.)

   (d) Show that $(\mathbb{Z}/n\mathbb{Z})^\times$ is the collection of residue classes whose representatives are relatively prime to n.

   (e) Write down the multiplication table for $(\mathbb{Z}/12\mathbb{Z})^\times$.

2. Let $\langle A, *_A\rangle$ and $\langle B, *_B\rangle$ be groups. Define the binary operation $*$ on $A \times B$ by $(a, b) * (c, d) = (a *_A c, b *_B d)$.

   (a) Show that $\langle A \times B, *\rangle$ is a group. The group $A \times B$ is called the ***direct product*** of A and B.

   (b) Show that $A \times B$ is abelian if and only if A and B are abelian.

3. Let G be a group, and $a, b \in \mathbb{Z}$. Let $x \in G$

   (a) Show $(x^a)^{-1} = x^{-a}$

   (b) Show $x^{a+b} = x^a x^b$ and $(x^a)^b = x^{ab}$.