Math 421 - Quiz 10
November 10, 2022

Name: Solutions

1. Let A and B be groups and set $G = A \times B$. Define the subgroup $H = \{(1, x) \mid x \in B\} \leq G$.

   (a) [*2 points*] Show that $H \trianglelefteq G$. (You don't need to show that it's a subgroup – just that it's normal.)

   Let $(a,b) \in G$, $(1,x) \in H$. Then

   $$(a,b)(1,x)(a,b)^{-1} = (a,b)(1,x)(a^{-1},b^{-1})$$
   $$= (aa^{-1}, bxb^{-1})$$
   $$= (1, bxb^{-1}) \in H$$

   so $gHg^{-1} \subseteq H$, so $H \trianglelefteq G$.

   (b) [*3 points*] Show that $G/H \cong A$. (Hint: 1st Isomorphism Theorem!)

   Define $\varphi : G \longrightarrow A$ by

   $$\varphi(a,b) = a.$$

   This is a homomorphism:

   $$\varphi((a,b)(a',b')) = \varphi(aa', bb') = aa'$$
   $$= \varphi(a,b)\varphi(a',b').$$

   It is surjective since for $a \in A$, $\varphi(a,1) = a$.

   $\ker \varphi = \{(a,b) \mid a = 1\} = H.$

   Thus, the 1st Isom. Thm. says

   $$G/H \cong \operatorname{im} \varphi = A.$$