Math 523 - Problem Set 1
Due Wednesday, Feb 2

1. Let A be an R-algebra. Show that A is finitely generated (as an R-algebra) if and only if there is a surjective R-algebra homomorphism
$$f : R[x_1, x_2, \ldots x_n] \twoheadrightarrow A$$
for some $n > 0$.

2. Prove that the following conditions on a module M over a ring R are equivalent.
   (i) M is Noetherian.
   (ii) Every ascending chain of submodules terminates.
   (iii) Every nonempty set Σ of submodules of M contains an element that is maximal under inclusion[1].

3. Let R be a Noetherian ring, and $I \subseteq R$ an ideal. A prime P containing I is ***minimal over*** I if there is no prime Q such that $I \subset Q \subsetneq P$. Show that there are only finitely many primes minimal over I . [2]

4. Let R be a ring with nilradical $\mathfrak{N}$. Show that the natural map $f : \mathrm{Spec}(R/\mathfrak{N}) \to \mathrm{Spec}(R)$ is a homeomorphism. That is, find an inverse and show that both f and f^{-1} send closed sets to closed sets.

5. Let $R = \mathbb{C}[x, y]$.
   (a) Fix a point $p \in \mathbb{C}^2$. Use the evaluation map $ev_p : R \to \mathbb{C}$, which sends f to $f(p)$, to show that the set of polynomials vanishing at p is a maximal ideal m_p of R.
   (b) Find generators for m_p. Can this ideal be principal? (Hint: R is a UFD.)
   (c) Show that the map $\phi : \mathbb{C}^2 \to \mathrm{Spec}\, R$ sending p to m_p is injective. Is it surjective?

6. Let $R = \mathbb{Z}[x, y]/(x^6, y^8, 24)$. Describe $\mathrm{Spec}\, R$.

7. If $I \subset R$ is an ideal, show that $\sqrt{I}$ is also an ideal in R, and show that it is radical. That is, $\sqrt{\sqrt{I}} = \sqrt{I}$.

8. Let k be a field and $R = k[x_1, \ldots, x_n]$. Check the following claims made in class regarding algebraic sets.
   (a) Let $f_1, \ldots, f_m \in R$ and $I = (f_1, \ldots, f_m)$. Show that
   $$Z(f_1, \ldots, f_m) = Z(I) = Z(\sqrt{I}).$$
   (b) If $S \subseteq S' \subseteq R$, then $Z(S') \subseteq Z(S)$.
   (c) If $X \subseteq k^n$ is any subset, then $I(X)$ is a radical ideal.

[1] This means that there is some element of Σ not properly contained in any other submodule in Σ.

[2] Hint: Assume not. Then there is an ideal I maximal among ideals for which it fails. This ideal can't be prime, so there are $f, g \in R - I$ such that $fg \in I$. Show that any prime minimal over I is minimal over (I, f) or (I, g).