Math 523 - Problem Set 2
Due Wednesday, Feb 16

1. Let $\phi : R \to S$ be a homomorphism of rings. Verify the claims made in class regarding the relationships between expansion and contraction of ideals:
   (a) $(J \cap R)S \subseteq J$ for all ideals $J \subseteq S$.
   (b) $I \subseteq (IS) \cap R$ for all ideals $I \subseteq R$.
   (c) For each (a) and (b), give an example in which the inclusion is proper.

2. Let $U \subseteq R$ be a multiplicative set.
   (a) Give an example to show that the natural map $R \to U^{-1}R$ is not always injective.
   (b) Give necessary and sufficient conditions for R to be a subring of $U^{-1}R$ (via the natural map).

3. Show that localization at U is a indeed a functor from R-modules to $U^{-1}R$-modules as follows:
   (a) If $\phi : M \to N$ is an R-module homomorphism, describe the natural map

   $$U^{-1}\Phi : U^{-1}M \to U^{-1}N$$

   and check that it is a well-defined $U^{-1}R$-module homomorphism.
   (b) If M is an R-module, show that $U^{-1}id_M = id_{U^{-1}M}$.
   (c) For R-module homomorphisms $\phi : M \to N$ and $\psi : N \to L$, show that

   $$U^{-1}(\psi \circ \phi) = (U^{-1}\psi) \circ (U^{-1}\phi).$$

4. Show that the natural map

   $$f : \operatorname{Spec}(U^{-1}R) \to \operatorname{Spec} R$$

   is a homeomorphism onto its image $Y = \{P \mid U \cap P = \emptyset\}$.[1]

5. If M is an R-module, use the universal property to verify that $R \otimes_R M \cong M$.

6. (a) Finish the proof from class of the claim that the tensor product is right exact.
   (b) Give an example to show that it is not necessarily left exact.[2]

7. (a) If I and J are ideals of R, show that $R/I \otimes_R R/J \cong R/(I+J)$.[3]
   (b) Describe $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n)$ for each pair $m, n \in \mathbb{Z}$.

8. Let $\phi : \mathbb{C}[t] \to \mathbb{C}[x, y, t]/(xy - t)$ be the inclusion map. Describe as explicitly as possible the fibers of the induced map of Spec.

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[1] The closed sets of Y are those of the form $C \cap Y$, where C is closed in $\operatorname{Spec} R$.

[2] Hint: Take $M = R/I$ and consider a sequence $0 \to R \to R \to R/(f)$, where $f \in I$ is chosen appropriately.

[3] Hint: Use right-exactness of tensor product.