Math 523 - Problem Set 3
Due Wednesday, March 2

1. Let $k = \mathbb{Z}/(2)$ and $R = k[x,y]/(x,y)^2$. Show that (x,y) is the union of three smaller ideals. Why doesn't this contradict prime avoidance?

2. Let k be an infinite field. Show that (x,y) in $k[x,y]$ is contained in an infinite union of primes P_i such that no P_i contains (x,y). Why doesn't this contradict prime avoidance?

3. Let M be a finitely generated $\mathbb{Z}$-module (i.e. abelian group). Describe the set of associated primes of M in terms of the usual structure theory for finitely generated abelian groups.

4. Let $R = \mathbb{C}[x,y,z]/(x^2 - yz)$. Let X, Y, Z be the images of x, y, z in R. Find minimal primary decompositions of (X) and (Y) in R. In each case, list the corresponding associated primes.

5. Let $R = \mathbb{C}[x,y,z]$. Let $I = (x^2yz, z^2)$. Find the support and associated primes of R/I, and give a minimal primary decomposition of I.

6. Let R be a ring and A an $n \times n$ matrix with entries in R, $\operatorname{adj}(A)$ the adjugate matrix of A, and $I_{n\times n}$ the $n \times n$ identity matrix. In this exercise we will prove the following matrix equation (which is a fact from linear algebra when R is a field):

$$\operatorname{adj}(A)A = \det(A)I_{n\times n}.$$

   (a) Show that R is isomorphic to the quotient of a polynomial ring $S = \mathbb{Z}[x_1, x_2, \ldots]$ (in possibly infinitely many variables).

   (b) Show that to prove the matrix equation for R it suffices to prove it for S.

   (c) Show that in order to prove the matrix equation over S, it suffices to prove it over its fraction field, and conclude the equation from the known linear algebra fact.

7. Let k be a field and $R = k[t]/(t^2)$. Set

$$p(x) = tx^3 + tx^2 - x^2 - x \in R[x].$$

   (a) Show that $S = R[x]/(p)$ is a free R module of rank 2.

   (b) We will prove a theorem in class that says S is a free module if and only if (p) is generated by a monic polynomial. Clearly p is not monic. How do you reconcile these two facts?

8. Let R be a local ring, with maximal ideal m. Let $I \subset R$ be an ideal, and suppose $x \in m$ is a nonzerodivisor on R/I.

   (a) Show that if I has a minimal set of generators, then it maps to a minimal generating set for the image of I in $R/(x)$.[1]

[1] Hint: Wait to do this problem until we've stated Nakayama's Lemma.

(b) Give an example to show that (a) can fail if x is a zerodivisor on R/I.

9. **(Fun/optional:)** Let k be a field and $R = k[x_1, \ldots, x_n]$. An ideal $I \subseteq R$ is a ***monomial ideal*** if it can be generated by monomials (i.e. elements of the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, for $a_i \geq 0$). Which monomial ideals are prime? primary? radical?