Math 523 - Problem Set 4
Due Wednesday, March 23

1. Let $R \subset S \subset T$ be rings.
   (a) If S is module-finite over R and T is module-finite over S, prove that T is module-finite over R.
   (b) If S is integral over R and T is integral over S, prove that T is integral over R.

2. Let $\mathbb{F}_2$ be the field with two elements, and $\mathbb{F}_2^X$ the ring of $\mathbb{F}_2$-valued functions on an infinite set X. Prove that $\mathbb{F}_2^X$ is integral over $\mathbb{F}_2$ and compute its dimension.

3. For each $n \in \mathbb{Z}$, find the integral closure R of $\mathbb{Z}[\sqrt{n}]$ as follows:
   (a) Show that you can reduce to the case where n is square-free.
   (b) Show that R is the integral closure of $\mathbb{Z}$ in the field $\mathbb{Q}[\sqrt{n}]$, and the minimal polynomial of $\alpha = a + b\sqrt{n}$ (with $a, b \in \mathbb{Q}$) is

   $$m(x) = x^2 - (2a)x + (a^2 - b^2 n).$$

   Conclude $\alpha \in R$ if and only if the coefficients of m are integers.
   (c) Show that if $\alpha \in R$, then $a \in \frac{1}{2}\mathbb{Z}$. If $a = 0$, show $\alpha \in R$ if and only if $b \in \mathbb{Z}$. If $a = 1/2$ and $\alpha \in R$, show that $b \in \frac{1}{2}\mathbb{Z}$. Thus, subtracting a multiple of $\sqrt{n}$, we can assume $b = 0$ or $1/2$.
   (d) Conclude that $R = \mathbb{Z}[1/2 + 1/2\sqrt{n}]$ if $n \equiv 1 \pmod 4$, and $\mathbb{Z}[\sqrt{n}]$ otherwise.

4. Let $R \subseteq S$ be integral domains. Assume there exists a map $S \to R$ which is the identity on R.
   (a) Show that if S is normal[1], then R is normal.
   (b) Let k be a field and $R = k[x^2, xy, y^2] \subseteq k[x, y]$. Show that R is normal, but not a *UFD*.

5. Let k be a field and $R = k[x(1-x), y, xy] \subseteq k[x, y] = S$.
   (a) Show that this is an integral extension of integral domains.[2]
   (b) Show that $Q = (1-x, y) \in \operatorname{Spec} S$ contracts to $P = (x(1-x), y, xy)$, which is maximal in R.
   (c) Show that $(x) \in \operatorname{Spec} S$ contracts to $P_0 = (x(1-x), xy)$.
   (d) Show that no prime Q_0 contained in Q contracts to P_0. Why doesn't this contradict the Going Down Theorem?

6. Let $R = \mathbb{C}[x, y]/(y^2 - x^2(x+1))$, which corresponds to the nodal curve we saw in class. Set $t = y/x$.

[1]Remember that an integral domain is ***normal*** if it is its own integral closure in its field of fractions
[2]Hint: Let $f(t) = t^2 - t + x(1-x)$.

(a) Show that $R[t] = \mathbb{C}[t]$. Conclude that $R[t]$ is the normalization of R.

(b) Consider the corresponding normalization map $\phi : \operatorname{Spec}(\mathbb{C}[t]) \to \operatorname{Spec}(R)$. Show that the point $(x, y) \in \operatorname{Spec}(R)$ (corresponding to the node at the origin) has exactly two points in its fiber. (In fact, it's an isomorphism away from that point!)