Math 523 - Problem Set 5
Due Wednesday, April 13

1. Let R be a ring and M an R-modules. Show that the following three statements are equivalent.

   i M is flat.

   ii $\operatorname{Tor}_i^R(M, N) = 0$ for all R-modules N and all $i > 0$.

   iii $\operatorname{Tor}_1^R(M, N) = 0$ for all R-modules N.

2. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. If $\{i, j, k\} = \{1, 2, 3\}$, for which values of i and j is the following statement true: If M_i and M_j are flat, then M_k is flat?

3. Calculate $\operatorname{Tor}_i^R(M, N)$ for all $i > 0$ in the following cases.

   (a) $R = \mathbb{C}[x, y]$, $M = R/(x)$, $N = R/(xy)$.

   (b) $R = \mathbb{Z}$, $M = \mathbb{Z}/(m)$, $N = \mathbb{Z}/(n)$.

   (c) $R = \mathbb{C}[x, y, z]$, $M = R/(x, y)$, $N = R/(x, y, z)$.

4. Let M and N be R-modules, and S a flat R-algebra. Show that

$$S \otimes \operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^S(S \otimes M, S \otimes N).$$

   In particular, Tor commutes with localization.

5. Let R be a ring and $U \subset R$ multiplicatively closed. Define $S = U^{-1}R$.

   (a) Show that for any prime $P \subset S$, the codimension of P in S is equal to the codimension of $P \cap R$ in R.

   (b) If $Q \subset R$ is prime, how does the codimension of Q compare to the codimension of QS in S?

6. Let R be an integral domain. Recall that R is ***normal*** if it is its own integral closure in its field of fractions. Show that the following are equivalent.[1]

   i. R is normal.

   ii. R_P is normal for all $P \subseteq R$ prime.

   iii. R_m is normal for all $m \subseteq R$ maximal.

7. Let R be Noetherian. Show that $\dim R[x, x^{-1}] = \dim R + 1$.

[1] Hint: We proved a proposition in the normalization section of the notes that will be helpful.