Math 523 - Problem Set 6
Due Wednesday, April 27

1. Let R be a ring and M an R-module. Let $\mathcal{J}$ be the filtration

$$M = M_0 \supset M_1 \supset \cdots .$$

   (a) Give an example to show that the function $\mathrm{in} : M \to \mathrm{gr}_{\mathcal{J}} M$ sending m to $\mathrm{in}(m)$ is not necessarily a group homomorphism.

   (b) Let $f, g \in M$. Suppose for each i, f is in M_i if and only if $g \in M_i$. Show that either $\mathrm{in}(\mathrm{in}(f) + \mathrm{in}(g)) = \mathrm{in}(f + g)$ or $\mathrm{in}(f) + \mathrm{in}(g) = 0$.

   (c) Suppose $M = R$ and that $\mathcal{J}$ is multiplicative, in the sense that $M_i M_j \subset M_{i+j}$. Show that either $\mathrm{in}(f)\,\mathrm{in}(g) = \mathrm{in}(fg)$ or $\mathrm{in}(f)\,\mathrm{in}(g) = 0$.

2. Let R be a Noetherian ring and $I \subset R$ an ideal. Let M be a finitely generated R-module.

   (a) Show there is a largest submodule $N \subset M$ such that N is annihilated by an element of the form $1 - r$ with $r \in I$.

   (b) Show that $\bigcap_{j=1}^{\infty} I^j M = N$.

3. Characterize the p-adic expansions in $\mathbb{Z}_p$ which are images of integers.

4. Let $R = k[x_1, \ldots, x_n]/I$, I some ideal. Let $m = (x_1, \ldots, x_n)$. Show that $\hat{R}_m = k[[x_1, \ldots, x_n]]/Ik[[x_1, \ldots, x_n]]$

5. Let $R = \mathbb{R}[x, y]/(y^2 - x^2 - x^3)$. Let $m = (x, y)$.

   (a) Verify that R is an integral domain.

   (b) Show that $\hat{R}_m$ is not an integral domain.[1]

   (c) Draw a picture of the curve in $\mathbb{R}^2$ cut out by $y^2 - x^2 - x^3 = 0$.

   (d) There is a sense in which $\mathrm{Spec}\,\hat{R}_m$ can be considered a very small neighborhood around $m \in \mathrm{Spec}\,R$. How is this reflected in your picture?

[1] Hint: Can you find a square root of $1 + x$?