let $R$ be a ring an $S$ an $R$-algebra.

**Def:** $s \in S$ is integral over $R$ if it is the zero of some monic polynomial in $R[x]$. If every element of $S$ is integral over $R$, then $S$ is integral over $R$.

We'll soon show that the set of elements integral over $R$ is a subalgebra of $S$, called the integral closure of $R$ in $S$ (or the normalization of $R$ in $S$). If $R$ is an integral domain, the integral closure (or normalization) of $R$ (w/out reference to an $R$-algebra) is the integral closure in its field of fractions.

Geometrically, normalizing a ring corresponds to "improving" the singularities of a variety, e.g. the normalization of a curve is always smooth!

**Def:** $S$ is finite over $R$ (or module-finite) if it is a finitely generated $R$-module (finite $\Rightarrow$ finitely generated as an $R$-algebra).

**Ex:** 1) $R[x]$ is a f.g. $R$-algebra, but is not finite or integral over $R$. 

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**Integrality**
2.) \( \frac{R[x]}{(x^2)} \) is finite and integral over \( R \).

3.) \( \mathbb{Q} \left[ \sqrt{2}, \sqrt{3}, \sqrt{2}, \ldots \right] \) is integral over \( \mathbb{Q} \), but not finite.

Finiteness is a stronger condition than integrality:

**Prop.** \( S \) is finite over \( R \) iff \( S \) is generated as an \( R \)-algebra by finitely many integral elements. (i.e. \( S = R[\alpha_1, \ldots, \alpha_n], \alpha_n \in S \))

\[ \text{image of } R \text{ generators integral over } R \]

**Pf:** Suppose \( S \) is finite over \( R \). If \( a \in S \), consider the map \( \varphi : S \rightarrow S \) defined \( s \mapsto as \). Then C-H says \( \varphi \) satisfies \( p(\varphi) = 0 \) for some monic polynomial \( p \). Thus \( p(a)S = 0 \) so \( p(a) = 0 \).

Conversely, suppose \( S = R[\alpha_1, \ldots, \alpha_n], \alpha_i \) integral over \( R \). Let \( S' = R[\alpha_1, \ldots, \alpha_{n-1}] \subseteq S \). By induction, we assume \( S' \) is finite over \( R \), generated by \( \{s_i\} \).

\( \alpha_n \) is integral over \( R \) and thus over \( S' \). Let \( p(x) \) be a monic polynomial over \( S' \) (or \( R \)) s.t. \( p(\alpha_n) = 0 \).

Then we have a map
\[
S'[x] \rightarrow S'[\alpha_n] = S
\]
\( x \mapsto \alpha_n \)
whose kernel contains \( p \).
So \( S'[x] \rightarrow S \cong S'[x]/I \)

where \( I \) is the whole kernel of \( S'[x] \rightarrow S \).

Thus, by the prop in the previous section, \( S \) is finite over \( S' \), generated by a finite set \( \{t_i\} \). Thus, \( S \) is generated as an \( R \)-module by \( \{s_it_j\} \). (Exercise!) \( \Box \)

Another application of C-H gives us a criterion for when an element is integral over a ring.

**Prop:** If \( S \) is an \( R \)-algebra and \( s \in S \), then \( s \) is integral over \( R \) iff \( \exists \) an \( S \)-module \( N \) and a f.g. \( R \)-submodule \( M \subseteq N \) not annihilated by any nonzero element of \( S \) s.t. \( sM \subseteq M \).

**Cor:** \( s \in S \) is integral over \( R \) \( \iff \) \( R[s] \) is finite over \( R \).

**Pf of Cor:** \( (\Rightarrow) \) by previous prop.

\( (\Leftarrow) \) Set \( M = N = R[s] \), and apply prop. \( \Box \)

**Pf of prop:** \( (\Rightarrow) \) Assume \( s \) is integral over \( R \). Take \( N = S \). Then \( M = R[s] \subseteq S \). Thus, \( M \cong R[x]/I \), where \( I \) contains some monic polynomial \( s \) satisfies, so \( M \) is finite over \( R \).
(⇒) Given \(M \in N\) modules as described, let \(\Phi: M \to M\) be a finite \(S\)-module multiplication by \(s\). Then we can apply C-H w/ \(I = R\), and we get a monic polynomial \(p(x) \in R[x]\) s.t. \(p(s)M = 0\). But \(M\) is not annihilated by nonzero elts of \(S\), so \(p(s) = 0\). Thus, \(s\) is integral over \(R\). □

As mentioned earlier, integral elements over \(R\) in \(S\) form a subalgebra. In particular, if \(s_1, \ldots, s_n\) are integral over \(R\), so is \(R[s_1, \ldots, s_n]\):

**Thus:** Let \(S\) be an \(R\)-algebra. The set of elements of \(S\) integral over \(R\) is a subalgebra of \(S\).

**Pf:** Suppose \(a, b \in S\) are integral over \(R\). WTS \(a - b, ab\) are as well.

\(R[a, b]\) is finite over \(R\). Let \(S = ab\) or \(a - b\), \(N = S\), \(M = R[a, b]\). \(M\) is not annihilated by any element of \(S\), and \(sM \subseteq M\). Thus, \(s\) is integral over \(R\) (by prop). □

**Field extensions + Nullstellensatz revisited**

we need to apply all of this to fields in order to finally finish the proof of the Nullstellensatz. Remember we need the following:
Thm: If \( k \) is algebraically closed, the maximal ideals of \( k[x_1, \ldots, x_n] \) are all of the form \((x_1-a_1, \ldots, x_n-a_n)\), \( a_i \in k \).

Recall that we know all of these ideals are maximal. We just need to show these are exactly the maximal ideals. First we recall some field theory:

Suppose \( K \subseteq L \) are fields, \( v_1, \ldots, v_n \in L \). Then \( K(v_1, \ldots, v_n) \) is the smallest subfield of \( L \) containing \( K \) and each \( v_i \). Equivalently, it's the field of fractions of \( K[v_1, \ldots, v_n] \).

Def: \( L \) is a **finitely generated field extension** of \( K \) if \( L = K(v_1, \ldots, v_n) \) for some \( v_1, \ldots, v_n \in L \). \( L \) is an **algebraic extension** if all elements of \( L \) are algebraic over \( K \), i.e., satisfy a polynomial over \( K \).

Ex: \( \mathbb{Q} [\sqrt{5}] = \mathbb{Q} (\sqrt{5}) \) since \( \sqrt{5} \left( \frac{\sqrt{5}}{5} \right) = 1 \). This is an algebraic extension of \( \mathbb{Q} \). It's also finite over \( \mathbb{Q} \) since it's generated by \( \sqrt{5} \).

Check: If \( K \subseteq L \) are fields, then the elements of \( L \) that are algebraic over \( K \) form a subfield.

Claim: \( k(x) \) is not a finitely generated \( k \)-algebra.
Suppose \( k(x) = k[v_1, \ldots, v_n] \).

Then \( \exists b \in k[x] \) s.t. \( bv_i \in k[x] \) for all \( v_i \), and choose \( c \in k[x] \) irreducible not dividing \( b \).

Write \( \frac{1}{c} \) as a \( k \)-linear combination of monomials in the \( v_i \)'s. \( \Rightarrow \exists N > 0 \) s.t. \( \frac{b^n}{c} \in k[x] \), a contradiction. \( \square \)

Claim: \( k[x] \) is its own integral closure in \( k(x) \).

Pf: Let \( z \in k(x) \) be integral over \( k[x] \). Then we have
\[
z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0, \quad a_i \in k[x].
\]

If \( z = \frac{f}{g} \), where \( f, g \in k[x] \) are relatively prime, then multiplying through by \( g^n \) we get:
\[
f^n + a_{n-1}f^{n-1}g + \ldots + a_0g^n = 0 \Rightarrow g \mid f^n \Rightarrow g \in k. \quad \square
\]

We need one more lemma before proving the Nullstellensatz.

Lemma: Let \( K \subseteq L \) be fields. If \( L \) is a f.g. \( K \) algebra, then \( L \) is finite (and thus algebraic) over \( K \).

Pf: Let \( L = K[v_1, \ldots, v_n] \). We'll prove by induction on \( n \).
If \( n=1 \), consider \( K[x] \to K[v_1] \)
\[
\begin{array}{ccc}
n & \mapsto & v_1 \\
\end{array}
\]

\( K[v_1] \) is a field, so \( K[v_1] \cong K[x]/(f) \), \( f \neq 0 \).

Thus, \( f(v_1) = 0 \Rightarrow v_1 \) is algebraic over \( K \Rightarrow K[v_1] \) is finite over \( K \).

Now assume the statement holds for extensions generated by \( n-1 \) elements. Then \( L = K(v_1)[v_2, \ldots, v_n] \) is finite \& algebraic over \( K(v_1) \).

**Case 1:** \( v_1 \) algebraic \( /K \). Then \( K(v_1) \) is algebraic \( /K \), so \( L \) is algebraic \( /K \) by transitivity of integrality (HW \#4).

So \( L \) is a \( K \)-algebra generated by finitely many integral elements, so it's finite over \( K \).

**Case 2:** \( v_1 \) not algebraic over \( K \). Then

\[
K[x] \to K[v_1]
\]

has 0 kernel, so \( K(x) \cong K(v_1) \).

Each \( v_i \) satisfies some \( v_i^n + a_1 v_i^{n-1} + \cdots + a_n = 0 \), \( a_j \in K(v_i) \).

Set \( a \in K[v_i] \) to be the product of the denominators of the \( a_j \). Multiplying by \( a^n \), we get
\[(a v_i)^n + a a_1 (a v_i)^{n-1} + \ldots + a^n a_n = 0,\]

where all the coefficients are now in \( K[v_i] \).

Thus, \( a v_i \) is integral over \( K[v_i] \).

Take \( z \in L \) and \( N \gg 0 \) s.t. \( a^N z \in K[v_i][a v_2, a v_3, \ldots, a v_n] \), and is thus integral over \( K[v_i] \).

Find \( c \in K[v_i] \) relatively prime to \( a \), and set
\[ z = \frac{1}{c} \in K(v_i) \subseteq L. \]

Then for some \( N > 0 \), \( \frac{a^N}{c} \in K(v_i) \) is integral over \( K[v_i] \), so \( \frac{a^N}{c} \in K[v_i] \), which is a contradiction since \( a \) and \( c \) are rel. prime. \( \square \)

Now we finish the Nullstellensatz:

**Theorem:** If \( k = \bar{k} \), and \( m \subseteq k[x_1, \ldots, x_n] = R \) is maximal, then \( m = (x_i - a_i, \ldots, x_n - a_n) \), some \( a_i \in k \).

**Pf:** Let \( L = R/m \). \( L \) is a field and \( k \subseteq L \).

\( L \) is f.g. over \( k \), so it's algebraic over \( k \), i.e. if \( z \in L \), then \( f(z) = 0 \), some \( f \in k[x] \).
$k$ is algebraically closed, so $z \in k$. Thus $L = k$.

Thus, for all $x_i$, there is some $a_i \in k$ s.t. $\overline{x_i} = \overline{a_i}$ in $L$, so

$\Rightarrow x_i - a_i \in m$.

$\Rightarrow (x_1 - a_1, \ldots, x_n - a_n) \subseteq m$, but is maximal, so they are equal. ∎