Math 524 - Problem Set 2
Due Wednesday, Feb 23

Let X be a topological space.

1. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Define the ***direct sum*** of $\mathcal{F}$ and $\mathcal{G}$, denoted $\mathcal{F} \oplus \mathcal{G}$, to be the presheaf which takes U to $\mathcal{F}(U) \oplus \mathcal{G}(U)$. Show that $\mathcal{F} \oplus \mathcal{G}$ is a sheaf.

2. Let k be a field, and $\mathcal{F}$ the sheaf of locally constant k-valued functions. Let $P, Q \in X$, and define $\mathcal{G} = k^{(P)} \oplus k^{(Q)}$, where $k^{(P)}$ and $k^{(Q)}$ are skyscraper sheaves with stalk k at P and Q, respectively. Define $\phi : \mathcal{F} \to \mathcal{G}$ by $\phi(U)(f) = (f(P), f(Q))$, where $f(P) = 0$ if $P \notin U$.

   (a) Verify that ϕ is a morphism of sheaves.

   (b) Show that ϕ is surjective.

   (c) Show that the map on sections $\phi(U)$ is not always surjective.

   (d) Describe the kernel of ϕ.

3. Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective and has kernel $\mathcal{F}'$.[1] Thus, there is an exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$

4. Let $\mathcal{F}$ be a sheaf on X, and $s \in \mathcal{F}(U)$ a section over an open set U. The ***support*** of s, denoted $\operatorname{Supp} s$ is defined to be $\{P \in U | s_P \neq 0\}$, where s_P is the germ of s in the stalk $\mathcal{F}_P$. Show that $\operatorname{Supp} s$ is a closed subset of U.

5. Let R be a ring. Recall that we defined
$$\mathcal{O}_{\operatorname{Spec} R}(U) = \varprojlim_{D(b) \subseteq U} R_b.$$

   (a) Let $a \in R$. Verify the claim that $\mathcal{O}(D(a)) = R_a$. (In particular, $\mathcal{O}(\operatorname{Spec} R) = \mathcal{O}(D(1)) = R$.)

   (b) Prove that $\mathcal{O}$ as defined above is actually a sheaf.[2]

6. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let U be the complement of the origin in $\mathbb{A}^n_{\mathbb{C}}$. That is, $U = \operatorname{Spec} R \setminus \{(x_1, x_2, \ldots, x_n)\}$. What is $\mathcal{O}(U)$?[3] Note that your answer may depend on n.

[1] You can use the facts from problem II1.2 in Hartshorne that surjectivity and injectivity can be checked on stalks.

[2] You can use the fact we discussed in class that it suffices to show that it satisfies the presheaf and sheaf axioms on a basis.

[3] Hint: Remember that we gave a nice description of $\mathcal{O}(U)$ when R is an integral domain.

7. Recall that a scheme X is ***reduced*** if $\mathcal{O}(U)$ has no nilpotents for each open $U \subset X$. Show that X is reduced if and only if for every $P \in X$ the stalk $\mathcal{O}_P$ has no nilpotents.[4]

8. Let A and B be rings, $\varphi : A \to B$ a ring homomorphism, and set $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Let $f : Y \to X$ be the morphism induced by φ.

   (a) Show that $f \in A$ is nilpotent if and only if $D(f) \subseteq X$ is empty.

   (b) Show that φ is injective if and only if $f^{\#} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show that in this case f is dominant (i.e. $f(Y)$ is dense in X).

   (c) Show that φ is surjective if and only if f is a homeomorphism of Y onto a closed subset of X and $f^{\#}$ is surjective.

[4]Hint: First show that you can reduce to the affine case.