# Math 524 - Problem Set 3
# Due Wednesday, March 16

1. Let $k$ be a field and $X$ a scheme. Show that if a morphism $f : X \to \operatorname{Spec} k$ is finite, then $X$ consists of finitely many points.

2. Let $A \to B$ be a morphism of rings. Recall that we defined $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$. Show that we have the following isomorphism (which I assumed without proof in class):
$$\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B.$$

3. Let $X$ be a scheme.
   (a) Show that there is a unique morphism $X \to \operatorname{Spec} \mathbb{Z}$.
   (b) Show there is a unique morphism from the zero ring to $X$.[1]

4. Let $X$ be an integral scheme and $Z \in X$ the generic point
   (a) Show that the local ring $\mathcal{O}_Z$ is a field. This is called the ***function field*** of $X$ and is denoted $K(X)$.
   (b) Show that if $U = \operatorname{Spec} A$ is any open affine in $X$, then $K(X)$ is isomorphic to the field of fractions of $A$.

5. Choose (at least) two of the following facts about morphisms of finite type to prove.
   (a) Closed immersions are of finite type.
   (b) Quasi-compact[2] open immersions are of finite type.
   (c) A composition of two morphisms of finite type is of finite type.
   (d) Morphisms of finite type are stable under base extension.

6. Let $X$ and $Y$ be schemes over an algebraically closed field $k$. Show that there is a one-to-one correspondence between closed points of $X \times_k Y$ and pairs $(x, y)$ where $x \in X$ and $y \in Y$ are closed points.

7. Show that finite morphisms are proper.

8. Let $X$ be separated over an affine scheme $S$. Show that the intersection of any two affine open subsets of $X$ is affine. (Challenge/optional: Show that this fails if $X$ is not separated.)

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[1] In the zero ring, $0 = 1$.

[2] $f : X \to Y$ is ***quasi-compact*** if there is a cover of $Y$ by open affines so that their preimages in $X$ are quasi-compact. Equivalently, if the preimage of every open affine is quasi-compact.