Math 524 - Problem Set 4
Due Wednesday, April 6

1. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on a scheme X.

   (a) For any open $U \subseteq X$, show that the set $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ of morphisms of the restricted sheaves has the natural structure of an abelian group.

   (b) Show that the presheaf $U \mapsto \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf. It's denoted $\mathscr{H}om(\mathcal{F}, \mathcal{G})$, and is called "sheaf hom."

2. Let X be a scheme and $\mathcal{E}$ a locally free sheaf of finite rank n. Define the ***dual*** of $\mathcal{E}$, denoted $\check{\mathcal{E}}$, to be the sheaf $\mathscr{H}om(\mathcal{E}, \mathcal{O}_X)$

   (a) Show that the stalk $\mathcal{E}_x$ at each point $x \in X$ is a free $\mathcal{O}_x$-module of rank n.

   (b) Describe the natural morphism $f : \mathcal{E} \to (\check{\mathcal{E}})\check{}$ by defining it on open sets. Make sure that your maps are compatible with restriction.

   (c) Show that f is an isomorphism.

3. Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf.

   (a) Suppose the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$. Show that there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

   (b) Show that the converse of 2(a) holds. That is, $\mathcal{F}$ is locally free if and only if $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module for every $x \in X$.

   (c) If $\mathcal{F}$ is invertible (locally free of rank one), show that $\mathcal{F} \otimes \check{\mathcal{F}} \cong \mathcal{O}_X$.[1]

   (d) Challenge/optional (requires additional knowledge about modules over local rings): Show that $\mathcal{F}$ is invertible if and only if there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$.

4. Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ a coherent sheaf on X. Recall that we gave an example in class showing that $f_*\mathcal{F}$ is not necessarily coherent, even if X and Y are both affine. Show that if X and Y are Noetherian and f is finite, then $f_*\mathcal{F}$ is coherent.

5. Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf on X. Recall that the residue field at a point $x \in X$ is $k(x) = \mathcal{O}_x/m_x$. Define a function

$$\varphi(x) = \dim_{k(x)}(\mathcal{F}_x \otimes_{\mathcal{O}_x} k(x)).$$

   Use Nakayama's Lemma to prove the following results about φ.

[1] Hint: Give a map on each open set, and show that the maps are compatible with restriction so that it defines a morphism. Then show that the morphism induces an isomorphism on stalks.

(a) φ is upper semi-continuous. That is, for any integer n, the set

$$\{x \in X \mid \varphi(x) \geq n\}$$

is closed.[2]

(b) If $\mathcal{F}$ is locally free and X connected, then φ is constant.

(c) If X is reduced and φ is constant, then $\mathcal{F}$ is locally free.

[2]Hint: Reduce to the affine case so that $\mathcal{F} \cong \tilde{M}$ and find a $k(x)$-basis at a point x. Show that there is some distinguished open $D(f)$ such that the same basis (after lifting to M) generates the $k(y)$-vector space at each $y \in D(f)$.