Math 524 - Problem Set 5
Due Wednesday, April 27

1. Let k be an algebraically closed field and X a closed subvariety of $\mathbb{P}^n_k$ which is nonsingular in codimension one (and thus satisfies the conditions need to define Weil divisors). For a divisor $D = \sum n_i Y_i$ on X, define the ***degree*** of D to be $\deg D = \sum n_i \deg Y_i$. Let V be an irreducible hypersurface in $\mathbb{P}^n$ which does not contain X and Y_i the irreducible components of $X \cap V$, all of which will have codimension one in X. For each Y_i, find an open affine U_i such that $Y_i \cap U_i$ is nonempty. Then Y_i is a principal divisor given by some f_i in the function field K. Set $n_i = v_{Y_i}(f_i)$ in U_i, and define $V.X$ to be $n_i Y_i \in \operatorname{Div} X$.

   (a) Show that $V \mapsto V.X$ determines a well-defined homomorphism from the subgroup H of $\operatorname{Div} \mathbb{P}^n$ consisting of divisors, none of whose components contain X, to $\operatorname{Div} X$.

   (b) If D is a principal divisor on $\mathbb{P}^n$, show that $D.X$ is principal on X. Thus, we get a homomorphism $\operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} X$.

2. Let $X = \operatorname{Spec} k[s^4, s^3t, st^3, t^4]$, k a field.

   (a) Show that X is Noetherian, separated, integral, and regular in codimension one.

   (b) Show that X is not normal.

   (c) Show that the principal divisor $D = (s^2t^2)$ is effective, but s^2t^2 is not regular on X.

3. Let X be a normal, separated, integral scheme with function field K. If D is a Weil divisor on X, we defined the associated sheaf $\mathcal{O}_X(D)$ by

$$U \mapsto \{f \in K^* \mid (f) + D_U \geq 0\}.$$

   (a) Check that $\mathcal{O}(D)$ is indeed a sheaf.

   (b) As mentioned in class, $\mathcal{O}_X(D)$ is invertible if and only if D is Cartier. As an example, let $R = \mathbb{C}[x, y, z]/(xy - z^2)$, $X = \operatorname{Spec} R$, and $Y \subset X$ the closed subscheme determined by the height one prime (y, z). We showed that the divisor Y is not Cartier. Verify that $\mathcal{O}_X(Y)$ is not invertible.

4. Let X be a scheme and $f : \mathcal{L} \to \mathcal{M}$ a surjective map of invertible sheaves on X. Show that f is an isomorphism.[1]

5. Let X be a scheme over a field k. Let $\mathcal{L}$ be an invertible sheaf on X. Suppose that $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ are two sets of global sections of $\mathcal{L}$ that span the same subspace $V \subseteq \Gamma(X, \mathcal{L})$ and which each generate each stalk of $\mathcal{L}$. Assume $n \leq m$. Show that the corresponding morphisms $\phi : X \to \mathbb{P}^n$ and $\psi : X \to \mathbb{P}^m$ differ by some linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is some linear subvariety of $\mathbb{P}^m$ of dimension $m - n - 1$.

[1] Hint: Stalks.