**Ringed spaces** (see Har II.2, Shaf IV.3.1)

To each ring $R$, we now have associated a topological space $\text{Spec} R$ and a sheaf of rings $\mathcal{O}$. We want to show that this is functorial, so we first need a category in which $(\text{Spec} R, \mathcal{O})$ is an object.

**Def:** A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf of rings $\mathcal{O}_X$ on $X$.

**Ex:** $(\text{Spec} R, \mathcal{O})$ is a ringed space

**Ex:** A topological space $X$ together w/ the sheaf of continuous $R$-valued functions is a ringed space. (i.e. not all ringed spaces look like $\text{Spec} R$).

**Def:** A **morphism of ringed spaces** from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ of sheaves of rings. (i.e. over each open $U \subseteq Y$ a ring homomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ that commutes with restriction maps.)
An important property of $(\text{Spec} R, \mathcal{O})$ is the fact that every stalk $\mathcal{O}_p$ is a local ring, which is reflected in the following:

**Def.** A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if for each $p \in X$, the stalk $\mathcal{O}_{X, p}$ is a local ring.

We want morphisms of loc. ringed spaces to respect the local ring structure on the stalks. That is:

**Def.** If $A$ and $B$ are local rings w/ max' $\text{ideal} m_A$ and $m_B$ respectively, then a homomorphism $\varphi: A \to B$ is a **local homomorphism** if $\varphi^{-1}(m_B) = m_A$, or equivalently $\varphi(m_A) \subseteq m_B$.

We know that a morphism $(f, f^\#)$ of ringed spaces induces maps between the stalks of $\mathcal{O}_Y$ and $f_*\mathcal{O}_X$, but we want maps $\mathcal{O}_{Y, f(p)} \to \mathcal{O}_{X, p}$.

Let $p \in X$. Then by definition we have

\[
\mathcal{O}_{Y, f(p)} \xrightarrow{f^\#} \varprojlim_{U \ni f(p)} \mathcal{O}_Y(U) \xrightarrow{\varpi f^{-1}} \lim_{V \ni f^{-1}(p)} \mathcal{O}_X(f^{-1}U) \xrightarrow{\varpi \mathcal{O}_X(V) = \mathcal{O}_{X, p}} \mathcal{O}_{X, p}
\]

Call this composition $f_p^\#$. 


This is well-defined since the \( f^{-1}(U) \) s.t. \( U \ni f(p) \) are a subset of all the open sets containing \( p \). So if \( (s, f^{-1}(U)) \sim (s', f^{-1}(U')) \) in the first direct limit then the will also be equivalent in the second.

**Def:** If \( X \) and \( Y \) are locally ringed spaces, then a morphism \( (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) of ringed spaces is a morphism of locally ringed spaces if for each \( p \in X \), \( f^\#_p : \mathcal{O}_Y, f(p) \to \mathcal{O}_X, p \) is a local homomorphism. (It's an isomorphism if \( f \) and \( f^\# \) are.)

\((\text{Spec } R, \mathcal{O})\) is clearly a locally ringed space, but we need to show that ring homomorphisms induce morphisms of locally ringed spaces.

**Theorem:** If \( \varphi : R \to S \) is a morphism of rings, then \( \varphi \) induces a natural morphism of locally ringed spaces

\[ (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), \]

where \( X = \text{Spec } S \), \( Y = \text{Spec } R \).

**Pf:** \( f : X \to Y \) is the map we've already defined:

\[ f(p) = \varphi^{-1}(p). \]

We first define \( f^\# \) on distinguished open sets by
\[ f^# : \mathcal{O}_Y(D(a)) \to f_* \mathcal{O}_X(D(a)) \]

\[ \mathcal{O}_X(f^{-1}(D(a))) \]

\[ \mathcal{O}_X(D(Y(a))) \]

Where \( \frac{r}{a} \to \frac{y(r)}{y(a)} \).

(Note that if \( y(a) = 0 \), then \( S_y(a) = 0 \) and this is the zero map.)

This uniquely extends to a morphism on each open \( U \subseteq Y \).

If \( P \in X = \text{Spec} Y \), then \( f(P) = y^{-1}(P) \in \text{Spec} X \), and the induced map on stalks is

\[ R_{y^{-1}(P)} \to S_P, \]

which is local by construction. \( \square \)

Conversely, all morphisms \( \text{Spec} B \to \text{Spec} A \) arise uniquely in this way. That is:

**Theorem:** If \( R \) and \( S \) are rings, then any morphism of locally ringed spaces \( f : \text{Spec} S \to \text{Spec} R \) is induced (uniquely) by a homomorphism \( y : R \to S \).
Thus, there is a one-to-one correspondence between such morphisms.

**Pf:** There is only one possible candidate for $\varphi$ (hence uniqueness): the induced map on global sections. So set $\varphi$ to be

$$f^\# : \Gamma(\text{spec } R, \mathcal{O}_{\text{spec } R}) \to \Gamma(\text{spec } S, \mathcal{O}_{\text{spec } S})$$

Thus, we just need to check that $(f, f^\#)$ is the map induced by $\varphi$.

We know that $\varphi$ is compatible with the map on stalks, so

$$\begin{array}{ccc}
R & \to & S \\
\downarrow & & \downarrow \\
R_{f(p)} & \overset{f^\#_p}{\to} & S_p
\end{array}$$

commutes. But $f^\#_p$ is local, which means that $(f^\#_p)^{-1}(P) = f(P)$. Commutativity of the diagram implies that $\varphi^{-1}(P) = f(P)$.

So the map $f$ is the one induced by $\varphi$.

Now maps $f^\# : R_a \to S_{\varphi(a)}$ over $D(a)$. 
are also compatible w/ $\Psi$, so they must be those induced by $\Psi$, so $f^*$ is induced by $\Psi$, and we're done. \( \square \)

**Remark:** In general, if $\Psi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, and we know $\Psi(u)$ for each element $u$ of a basis, then we can recover $\Psi$. (Use sheaf condition.)