**Hyperelliptic curves**

**Remark:** Recall that a curve is hyperelliptic if it has a linear system of dimension 1 and degree 2. The classical notation for such a linear system is $g_2^1$. More generally, a linear system of dimension $r$ and degree $d$ is called a $g_d^r$.

In the case of hyperelliptic curves, it turns out that any $g_2^1$ is unique:

**Prop:** Let $X$ be a hyperelliptic curve of genus $g \geq 2$. Then $X$ has a unique $g_2^1$.

**Pf:** Fix a $g_2^1$, $|V|$. Let $P + Q \in |V|$. Then $K_X$ is base-point free but not very ample, so let

$$ f : X \to |Pj^{-1}| $$

be the morphism defined by $|K_X|$.

Then $\dim |K_X - P| = \dim |K_X| - 1$, but $\dim |K_X - P - Q| = \dim |K_X| - 1$. Thus $Q$ is a base-point of $|K_X - P|$. $|K_X - P|$ is in one-to-one correspondence with the elements of $|K_X|$ in which $P$ appears.
Thus, $Q$ and $P$ appear in all of the same divisors of $|K_x|$, so $f(Q) = f(P)$.

Since $|P+Q|$ is infinite, there are infinitely many effective divisors linearly equivalent to $P+Q$, so $f$ cannot be birational (generically 1-to-1) onto its image.

Let $X' \subseteq P^{g-1}$ be the image of $f$ and $\mu \geq 2$ the degree of $f : X \to X'$.

Since $\deg K_x = 2g - 2$, we have $d\mu = 2g - 2$, so $d \leq g - 1$.

Now, $X'$ may be singular, so let $\tilde{X}'$ be the normalization of $X'$.

Let $|W|$ be the linear system on $\tilde{X}'$ corresponding to the map $\tilde{X}' \to X' \subseteq P^{g-1}$. Then, since it is birational onto its image, $\deg |W| = d$ and $\dim |W| = g - 1$.

Since $d \leq g - 1$, we thus must have $d = g - 1$ (see HW 1 #2a).

But the equality $\deg |W| = \dim |W|$ holds iff $d = 0$ (which it's not) or the genus of the curve $\tilde{X}'$ is 0.
Thus, \( \hat{X}' \cong \mathbb{P}^1 \), so the linear system \(|W|\) is the complete linear system of \( \text{deg } d = g-1 \), i.e. \(|(g-1)\mathbb{P}|\), for any point \( P \in \hat{X}' \), and \( \hat{X}' \to \mathbb{P}^{g-1} \) is an embedding.

In particular, \( \hat{X}' \to X' \) is an isomorphism, so \( X' \cong \mathbb{P}^1 \).

Back to \( f : X \to X' \), since \( d \mu = 2g-2 \) and \( d = g-1 \), we get \( \mu = 2 \), so \( f \) is the composition

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \subset \mathbb{P}^{g-1} \\
\downarrow{f_0} & & \ \ \downarrow \\
X' & \cong & \mathbb{P}^1 \\
\end{array}
\]

where \( f_0 \) is a 2:1 map onto \( \mathbb{P}^1 \). Since \( f(P) = f(Q) \), we know \( f_0(P) = f_0(Q) \), so the linear system corresponding to \( f_0 \) is \( |P+Q| \).

But \( |P+Q| \) was chosen to be any \( g_2 \), so it must be unique! □

**Remark:** If we look again at the composition

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{P}^{g-1} \\
\downarrow{\text{canonical}} & & \ \ \downarrow{\text{deg } g-1} \\
 & & \mathbb{P}^1 \\
\end{array}
\]

we see that any divisor in \(|K_X|\) will be the pull-back
of a hyperplane section, which in this case is just the sum of $g-1$ divisors from the $g'_2$.

On the other hand, since $\mathbb{P}^{g-1} \rightarrow \mathbb{P}^1$ corresponds to the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(g-1)|$, every deg $g-1$ effective divisor in $\mathbb{P}^1$ appears as a hyperplane section. Thus, every sum of $g-1$ divisors in $g'_2$ is in $|K_x|$.

so we can write $|K_x| = \sum_{i=1}^{g-1} g'_2$, to mean

$$K_x \sim D_1 + \ldots + D_{g-1}, \text{ any } D_i \in g'_2.$$  

**Clifford's Theorem**

We can use the above prop to prove Clifford's Theorem, which gives a bound on the dimension of a linear system.

Let $D$ be a divisor on a curve $X$. If $h^1(D) = h^0(K-D) = 0$, we know that $R-R$ gives us an explicit formula for $\dim |D|$, namely: $\dim |D| = \deg D - g$.

However, if $D$ is special i.e. $h^0(K_x-D) > 0$, this just gives us a lower bound, $\dim |D| > \deg D - g$.

Using basic properties of divisors, we showed on the
homework the upper bound \( \dim |D| \leq \deg D \).

For special divisors, we can show:

**Clifford's Theorem:** Let \( D \) be an effective special divisor on a curve \( X \). Then

\[
\dim |D| \leq \frac{1}{2} \deg D.
\]

Equality occurs if and only if either

- \( D = 0 \),
- \( D = K_x \), or
- \( X \) is hyperelliptic and \( D \) is a multiple of the unique \( g^1_2 \) on \( X \).

Before we prove this, we need the following:

**Lemma:** Let \( D, E \) be effective divisors on a curve \( X \). Then

\[
\dim |D| + \dim |E| \leq \dim |D + E|.
\]

**Pf:** Consider the function

\[
|D| \times |E| \to |D + E|
\]

defined \( (D', E') \mapsto D' + E' \).

This map is finite-to-one since for any \( F \in |D + E| \), there
are only finitely many ways to break it up as the sum of two divisors.

Moreover, the map is actually a morphism, since it corresponds to the (bilinear) natural map

\[ H^0(O(D)) \times H^0(O(E)) \to H^0(O(D+E)). \]

Thus, the dimension of the image is equal to

\[ \dim |D| + \dim |E|, \text{ which is thus at most } \dim |D+E|. \]

Now we can prove Clifford’s Theorem:

**Pf of Clifford’s Theorem:** If $D$ is effective and special, then $h^0(K_x - D) > 0$, so there is some choice of $K_x$ s.t. $K_x - D$ is effective.

Thus, the above lemma says

\[ \dim |D| + \dim |K_x - D| \leq \dim |K_x| = g - 1. \]

R-R tells us

\[ \dim |D| = \dim |K_x - D| + \deg D + 1 - g \]

Adding these together, we get

\[ 2 \dim |D| \leq \deg D \]
$\Rightarrow \dim |D| \leq \frac{1}{2} \deg D$. This gives the first part of the theorem. (Note that equality holds exactly when equality holds in *)

1. If $D = 0$, then $\dim |D| = 0 = \frac{1}{2} \deg D$.  
2. If $D = K$, then $\dim |K| = g-1 = \frac{1}{2}(2g-2) = \frac{1}{2} \deg K$.

3. If $X$ is hyperelliptic and $D$ is a multiple of the $g_j^1$, i.e. $D \sim D_1 + \cdots + D_r$, $D_i \in g_j^1$, then

$$\dim |D| \geq \Sigma \dim |D_i| = r$$

by the lemma,

but $\dim |D| \leq \frac{1}{2} \deg K = r$, so $\dim |D| = r = \frac{1}{2} \deg D$.

Conversely, assume $\dim |D| = \frac{1}{2} \deg D$ and $D \neq 0, K_x$. We want to show then that $X$ is hyperelliptic, and $D$ is a multiple of the $g_j^1$.

We'll prove this by induction on $\deg D$, which is even. Thus, the base case is $\deg D = 2$, in which case $\dim |D| = 1$, so $D$ is itself a $g_j^1$, and $X$ is hyperelliptic.

Now assume $\deg D \geq 4$.

Fix $E \in |K - D|$, and choose $P, Q \in X$ such that $P \notin \text{Supp } E$
but \( Q \notin \text{Supp} \mathcal{E} \).

\[
\text{since } \dim |D| = \frac{1}{2} \deg D \geq 2, \quad |D - P - Q| \neq \emptyset
\]

That is, \( \text{WLOG} \), we can replace \( D \) by some element of \( |D| \) that has both \( P, Q \in \text{Supp} \mathcal{D} \).

Let \( D' = D \cap E \), the largest divisor s.t. \( D - D' \) and \( E - D' \) are both effective. Then \( Q \notin \text{Supp} D' \), and \( P \in \text{Supp} D' \), so

\[
0 < \deg D' < \deg D.
\]

We have an exact sequence:

\[
0 \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(E)) \to H^0(\mathcal{O}(D + E - D'))
\]

Projectivizing, we get

\[
\dim |D| + \dim |E| \leq \dim |D + E - D'| + \dim |D'| \leq \dim |K| = g - 1
\]

\[
\dim |K - D| = \dim |K - D'|
\]

But \( \frac{1}{2} \deg D = \dim |D| \), so LHS = \( g - 1 \), by the equality in \( * \).

Thus, \( \dim |K - D'| + \dim |D'| = g - 1 \), so \( \dim |D'| = \frac{1}{2} \deg D' \) as well.
Thus, by the induction hypothesis, $X$ is hyperelliptic.

Now consider the linear system $|D| + (g - 1 - \dim |D|)g^1$. It has degree $\deg D + 2(g - 1 - \frac{1}{2} \deg D) = 2g - 2$, and dimension \geq \dim |D| + (g - 1 - \dim |D|) = g - 1,

so if we choose a representative divisor $A$, then R-R says $\dim |k - A| \geq (g - 1) + 0 + 1 - g = 0$, so $K \sim A$.

Thus, $|D| = r g^1$. $\square$