Hints for HW of Chapter 3.

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3.1.(c) $\operatorname{tr}(f(A)) = \sum_{i=1}^{n} f(\lambda_i)$.

3.5. For eigenspace of A, find all possible solutions of $(A-\lambda I)\vec{x} = 0$.

3.6. Compute $(A+\gamma I_m)\vec{x}_i = \cdots\cdots = (\lambda_i + \gamma)\vec{x}_i$

3.9. $|AB-\lambda I| = \cdots\cdots \quad \cdots\cdots = |BA-\lambda I|$.

(write I as $B^{-1}B$)

3.10. $A\vec{x} = \lambda\vec{x}$, take complex conjugate on both sides:

$A\bar{\vec{x}} = \lambda\bar{\vec{x}}$, then add the two equations.

3.11. Easy verification.

3.14. Modify the proof of Theorem 3.9 on p.106.

3.15. Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Can you find anothe matrix C, different from A s.t. C has two zeros as eigenvalues?

3.16 Apply thm 3.3 with $r=1$.

3.17. Thm 2.24 + Thm 3.3.

Note that $\dim(\text{Nullspace of } A-\lambda I) = \dim(S_A(\lambda))$.

3.18. Check eigenvalues.

3.19. Direct computation.

(Algebraic multiplicity $= m >$ geometric multiplicity $=1$)

3.20. (a) Try $\vec{x}$ and vectors that are orthogonal to $\vec{y}$.

(b) Multiply: $(I_m + \vec{x}\vec{y}^T)(I_m - c^{-1}\vec{x}\vec{y}^T) = \cdots = I_m$

use the form of c.

3.21. Easy. Try. (Eigenvalues of $f(A)$ are $f(\lambda_i)$)

3.22. Row sum is 1 $\Leftrightarrow$ $A\vec{1} = \vec{1}$.

Then use the fact that A is nonsingular.

3.23. (a) $(1+\lambda)^{-1} + (1+\lambda^{-1})^{-1} = \dfrac{1}{1+\lambda} + \dfrac{\lambda}{1+\lambda} = 1$.

(b) Thm 1.9 $A=C=D=I$.

3.25. Direct computation.

$$|A-\lambda I| = \lambda^2 - (\mathrm{tr} A)\lambda + |A|.$$

3.27. Special case of 3.20.

3.29. Follow 3.27.

(e) $|A| = \prod_{i=1}^{m} \lambda_i$.

3.33. $LHS = \mathrm{tr}(AA^T)$

write A as $X\Lambda X^T$.

3.34. Follow 3.29, find all eigenvalues of A.

A positive def. $\Leftrightarrow$ All eigenvalues are positive.

3.35. Use 3.33.

3.36. Take $B = \begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix}$ Find A?

3.39 (a) $A\vec{x} = \lambda\vec{x} \Rightarrow \quad 0 = A^k\vec{x} = \lambda^k\vec{x} \Rightarrow \quad \Rightarrow \lambda = 0.$

3.46. You need to show.

(i) $\max_{\vec{x}^T A\vec{x}=1} \frac{1}{\vec{x}^T\vec{x}} \le \lambda_1(A).$

(4)

(ii) The maximum can be obtained by taking $\vec{x} = \lambda_1^{-1/2} \vec{x}_1$.

3.52. (a) use the simultaneous diagonalization of A & B.

i.e., $\exists$ C nonsingular (p. 136) s.t.

$$\begin{cases} A = C C^T \\ B = C \Lambda C^T \end{cases}$$

$$\Rightarrow \quad |A+B| = |C(I+\Lambda)C^T| = \cdots$$

$$\left(|I+\Lambda| \geq |I| \text{ because } \Lambda \text{ has nonnegative diagonal entries.} \right)$$

(b) $$|A| = |B + (A-B)| \underset{(a)}{\geq} \cdots$$

3.55. $X = [\vec{x}_1, \cdots, \vec{x}_m]$

$$\left.\begin{aligned} A &= X \Lambda X^T \\ B &= X \Gamma X^T \end{aligned}\right\} \Rightarrow \left| \begin{aligned} A+B &= \cdots \\ AB &= \cdots \\ BA &= \cdots \end{aligned} \right.$$

3.57. $\mathrm{tr}(B^TAB) = \sum_{i=1}^{r} \lambda_i (B^TAB)$.

(a) For $\min_{B^TB=I_r} \mathrm{tr}(B^TAB) = \sum_{i=1}^{r} \lambda_{m-i+1}$.

"$\geq$" is due to Thm 3.19.

Then show that you can obtain this minimum.

Similarly for $\max_{B^TB=I_r} \mathrm{tr}(B^TAB) = \sum_{i=1}^{r} \lambda_i$

(b) Take a special B.

3.60. Use thm 3.32.