Hints for Chapter 4.

Math 463/663. ①
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4.1. Easy.

4.2. Definition & Theorem 3.27 on p. 131.

4.3. "$\Rightarrow$": $A\vec{x} = 0 \cdot \vec{x} \Rightarrow \cdots \; A^T A \vec{x} = 0 \Rightarrow \cdots$

"$\Leftarrow$": $A^T A \vec{x} = 0 \;(\vec{x} \neq 0) \Rightarrow \vec{x}^T A^T A \vec{x} = 0 \Leftrightarrow (A\vec{x})^T (A\vec{x}) = 0 \Rightarrow A\vec{x} = 0 \Rightarrow \cdots$

4.4. Let $m > 1$ and $n = 1$.

$A =$ a column vector and $B =$ a row vector.

$AB =$, the singular values correspond to square root of the eigenvalue of

$AB(AB)^T = ABB^TA^T =$

a number.

$\Rightarrow$ singular values: 0 and $\sqrt{\;}$.

$BA =$ is a scalar. So, singular value is $|\;|$.

IN general, $|\;| \neq \sqrt{\;}$ for example, one can choose

$\perp$.

4.5 $$B - \lambda I = \begin{pmatrix} -\lambda I_m & A \\ A^T & -\lambda I_n \end{pmatrix}$$

$$\begin{pmatrix} I_m & \lambda^{-1} A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} -\lambda I_m & A \\ A^T & -\lambda I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ \lambda^{-1} A^T & I_n \end{pmatrix} = \cdots = \begin{pmatrix} -\lambda I_m + \lambda^{-1} A A^T & 0 \\ 0 & -\lambda I_n \end{pmatrix}$$

So $$|B - \lambda I| = (-\lambda)^n \cdot |-\lambda I_m + \lambda^{-1} A A^T|$$

$$= (-\lambda)^n \cdot \lambda^{-m} \cdot |AA^T - \lambda^2 I_m| = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \lambda^2 = \cdots\cdots$$

4.6. $A = \| \Rightarrow AA^T = \| \Rightarrow \Rightarrow$ Eigenvalues. see Ex 3.20. on P.147.

4.7. Application of Corollary 4.1.1.

4.8. Use SVD decomposition:

$$\underset{m \times n}{A} = \underset{m \times r}{P} \underset{r \times r}{\Lambda} \underset{r \times n}{Q^T} = P \Lambda \underbrace{P^T P}_{= I_r} Q^T \Rightarrow \cdots\cdots$$

4.14 A, $\operatorname{rank}(A) = r$

Let eigenvalues be $\lambda_1 \cdots \lambda_r$, $\lambda_{r+1} = \cdots = \lambda_m = 0$.

$$\Lambda = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$$A = \underbrace{[\vec{x}_1 \cdots \vec{x}_r, \vec{x}_{r+1} \cdots \vec{x}_m]}_{X} \left[\begin{array}{ccc|ccc} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_r & & & \\ \hline & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right] X^T$$

$X = [B, C]$

first r columns.

$$= [B, C] \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B^T \\ C^T \end{bmatrix}$$

$= \cdots\cdots$

write $\Lambda_r = \sqrt{\Lambda_r} \cdot \sqrt{\Lambda_r}$.

$= \sqrt{\Lambda_r} \cdot \underbrace{B^T B}_{= I_r} \sqrt{\Lambda_r}$

4.19. Easy.

$AB = X \Lambda X^{-1}$.

If B is nonsingular, then $A = X \Lambda X^{-1} B^{-1}$.

then $\cdots\cdots$.

For conter examples try 2x2 matrizes, e.g. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ....

4.20: First find T, symmetric, positive def. s.t. $A = T^2$.

Then $AB = T^2 B = T\,(TBT)\,T^{-1}$

Spectral decomposition on TBT, ...

4.21. ① ~~A, B diagonalizable $\Rightarrow$ C diagonalizable~~ : trivial.

② $\Leftarrow$

Use the fact that C is diagonalizable iff there exist $(m+n)$ linearly indep. eigenvectors of C.

4.22. The same hint as 4.21.

4.25. (a) $A = \begin{pmatrix} 1 & & \\ & 0 & \\ & 0 & 0 \end{pmatrix}$

(b) $A = \cdots$ do something on the upper triangle part.

(c) Same as (b).

4.26. 7 forms. Try to find all of them.

4.27. (a) Easy

(b) Easy.

4.28 (a) $J_n(\lambda) = \lambda I_n + B_n$ with $B_n = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}_{n \times n}$.

$B_n^2 = \begin{bmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \\ 0 & & & & 0 \end{bmatrix}$, $\cdots$ $B_n^{k-1} = (0)$.

(b) Easy

(c) $A = X J X^{-1}$

$= X (D + B) X^{-1}$ ~ (b)

$= X D X^{-1} + X B X^{-1}$

$\Rightarrow$ $X B X^{-1}$ is Nilpotent.

4.32. Use Jordan decomposition.

$\text{rank}(A) = \#\{\lambda_i, \lambda_i \neq 0\}$ $\Leftrightarrow$ No $J_n(0), (n>1)$ exists!

$\text{rank}(A) = \text{rank}(J)$.

$\|$ $\Leftrightarrow$ $\cdots\cdots$

$\text{rank}(A^2) = \text{rank}(J^2)$.

4.34. $\left.\begin{array}{l} X^* A X = T_1 \\ X^* B X = T_2 \end{array}\right\} \Rightarrow \begin{array}{l} X^* A B X = \cdots \\ X^* B A X = \cdots\cdots \end{array}$

↑

Substract the two $\Rightarrow \cdots$

$\Rightarrow \quad C(A-B)C^T = \Lambda - I.$

$A-B$ positive def. iff $\lambda_i - 1 > 0$. $i=1,\dots m$.

On the other hand.

$$\begin{cases} (C^T)^{-1} A^{-1} C^{-1} = \Lambda^{-1} \\ (C^T)^{-1} B^{-1} C^{-1} = I \end{cases} \Rightarrow (C^T)^{-1}(B^{-1} - A^{-1}) C^{-1} = I - \Lambda^{-1}$$

So. $B^{-1} - A^{-1}$ is positive def. iff

$1 - \frac{1}{\lambda_i} > 0$ for all $i=1,\dots,m$.

4.47. A, B are simultaneously diagonalizable, i.e., $\exists\, C$ nonsingular,

$$\begin{cases} CAC^T = \Lambda \\ CBC^T = I \end{cases}$$

On the one hand, $C(A+B)C^T = \Lambda + I \Rightarrow A+B$ positive def.

iff $\lambda_i + 1 > 0$.

On the other hand, $(C^T)^{-1} B^{-1} C^{-1} = I$

$$\Rightarrow C A B^{-1} C^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

$$\Rightarrow \lambda_i (AB^{-1}) = \lambda_i$$

....

4.37. Follow example 4.11 on P177.

4.39. $tr(TT^*) = \sum_{i \leq j} |t_{ij}|^2 = \sum_{i<j} |t_{ij}|^2 + \underbrace{\sum_{i=1}^{m} |t_{ii}|^2}_{\parallel \atop \sum_{i=1}^{m} |\lambda_i|^2}$

So we only need to show that $\sum_{i \leq j} |t_{ij}|^2$ is uniquely defined.

"$\leq$"

NOT "<"

Let $X_1^* A X_1 = T_1$ and $X_2^* A X_2 = T_2$ be two Schur dec.

Then $tr(T_1 T_1^*) = \cdots \qquad = \cdots$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel ?$

$tr(T_2 T_2^*) = \cdots \qquad = \cdots$

4.41. Follow almost line-by-line the proof of theorem 4.18. You need to use Ex 4.21.

4.42. Easy for (a) & (b)

(c) Is B diagonalizable?

4.43. Apply Ex 4.41.

Note, $(i_1, \cdots, i_m)$ and $(j_1, \cdots j_m)$ are some specific permutations of $(1, \cdots, m)$.

4.44 (a) (b). Easy!

4.46. A, B are simultaneously diagonalizable. i.e. $\exists$ C nonsingular s.t.

$$\begin{cases} CAC^T = \Lambda \\ CBC^T = I. \end{cases}$$

(Textbook P. 136).

4.49.

(a) By the proof of Thm 3.29 on P134–P135, there exists a symmetric, non singular matrix T s.t. $B = T^2$.

$A = X \Lambda X^T$ ~ spectral decomposition, $X^T = X^{-1}$.

$\Rightarrow AB = X \Lambda X^T T^2$

$\Rightarrow T^{-1} A B T^{-1} = \ldots\ldots$

4.57: Try some asymmetric matrices.