Hints for HW of Chapter 5.

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5.2. $A^+ = (A^TA)^{-1}A^T$ if $\text{rank}(A) = n$.

5.3 Same as 5.2.

5.6. Follow example 5.2 on P. 209.

5.7. (a) Follow the same way as Ex 5.6 to find $(AA^T)^+$.

Then by Thm 5.3(g), $A^+ = (AA^T)^+ A^T$.

(b) $P_{R(X)} = XX^+$ ((5.7) on P209).

or use the (normalized) eigenvectors of AA^T as in the second part of Example 5.3 on P. 209–210.

Projection to the row space corresponds to the projection to column space of X'.

So. $X'(X')^+ = (X^+X)' \underset{\substack{5.1(a)\\ P202}}{=} X^+X$.

5.9. $\text{rank}(A) = 1 \Rightarrow A$ has SVD of the form: $A = \vec{p}\, d\, \vec{q}^{\,T}$.

$\vec{p}$: $m \times 1$

$\vec{q}$: $n \times 1$ $\Rightarrow A^+ = \vec{q}\, d^{-1} \vec{p}^{\,T} = \cdots\cdots$

d : scalar.

5.10.

(a) rank $(A) = 1$. Apply Ex 5.9.

(b) Is A idempotent? Apply Thm 5.5 (c).

(c) Apply Ex 5.9.

(d) " "

5.12: Easy. Use (5.1)–(5.4) on p 202.

5.16: AA^+ is the projection matrix to the column space of A. i.e. $R(A)$

$(AB)(AB)^+$ is " " " " AB. i.e. $R(AB)$

So $AA^+ = (AB)(AB)^+ \iff$ these two spaces are the same.

i.e. $R(A) = R(AB)$

① $R(AB) \subseteq R(A)$

② $R(A) \subseteq R(AB)$

$\forall \vec{x} \in R(A)$, $\exists \vec{y}$ s.t. $A\vec{y} = \vec{x}$.

Because B has full row rank, apply Thm 5.3 (i), $BB^+ = I_n$

$\Rightarrow A\vec{y} = A I_n \vec{y} = AB\underbrace{B^+\vec{y}} = \vec{x}$

$\Rightarrow \vec{x} \in R(AB)$

$\square$

5.13 B is pos. def. $\Rightarrow \exists T$ symmetric, nonsingular s.t.

$$B = T^2.$$

So we need to prove $(AT)(AT)'\left[(AT)(AT)'\right]^{+} \underbrace{A}_{(TT^{-1})} = A.$

Set $F = AT$. So we need to show

$$FF^{T}(FF^{T})^{+}FT^{-1} = A.$$

$\vdots$

5.11. They are all projection matrices. Easy to verify.

5.34. Use the SVD decomposition (see Thm 5.22).

5.35. The same as Ex 5.34.

5.38. ① $(AB)(B^{-}A^{-})(AB) = AB \underset{??}{\Rightarrow} (A^{-}ABB^{-})^{2} = A^{-}ABB^{-}$

② $(A^{-}ABB^{-})^{2} = A^{-}ABB^{-} \underset{??}{\Rightarrow} (AB)(B^{-}A^{-})(AB) = AB.$

5.39. (1) $ABA = A \overset{??}{\Rightarrow} (AB)^2 = AB$ and $\text{rank}(A) = \text{rank}(AB)$.

(2) "$\Leftarrow$"

$$\underbrace{\text{rank}(AB) = \text{rank}((AB)^2)}_{(AB)^2 = AB} = \text{rank}(ABAB) \le \text{rank}(ABA) \le \text{rank}(AB) \le \text{rank}(A)$$

$\Rightarrow \text{rank}(ABA) = \text{rank}(A)$. --- (*)

Clearly $R(ABA) \subseteq R(A)$. but (*) ensures that $\exists D$ invertible s.t.

$$ABA = AD.$$

$$\Rightarrow \underbrace{\underline{AB}\,ABA}_{=ABA} = \underline{AB}AD \quad \Rightarrow ABAD = AD$$

Now use the fact that D is nonsingular. $\square$

5.40. We need to show that

$$(PAQ)\, Q^{-1}A^{-}P^{-}\, (PAQ) = PAQ$$

Now use the properties (f) and (g) of thm 5.23 on p.227.

5.53. We need to show that $A'(AA')^L$ satisfies (5.1)–(5.4).

① (5.1)

$$A \cdot A'(AA')^L A = \cdots = A.$$

↑ Thm 5.27 (i.e. $A(A'A)^- A' = AA^+$)

② (5.2)

$$A'(AA')^L A \, A'(AA')^L = \cdots = A'(AA')^L.$$

↑ Thm 5.27

③ (5.3)

$AA'(AA')^L$ is symmetric? (Thm 5.28 (a))

④ (5.4)

$A'(AA')^L A$ is symmetric? (trivial).

5.55. As in Ex 5.53, verify all properties (5.1)–(5.4).