Hints for HW of Chapter 9.

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Compute:

9.2. $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, $\frac{\partial^2 f}{\partial x_1^2}$, $\frac{\partial^2 f}{\partial x_2^2}$, $\frac{\partial f^2}{\partial x_1 \partial x_2}$.

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

$$f(0+\vec{u}) = f(0) + \left(\frac{\partial}{\partial x_1} f(0)\cdot u_1 + \frac{\partial}{\partial x_2} f(0)\, u_2\right)$$
$$+ \left(\frac{\partial^2}{\partial x_1^2} f(0)\, u_1^2 + \frac{\partial^2}{\partial x_2^2} f(0)\cdot u_2^2 + 2\frac{\partial^2}{\partial x_1 \partial x_2} f(0)\, u_1 u_2\right)$$
$$+ r_2(0,\vec{u}).$$

9.3. $\vec{y}(\vec{x}) = \vec{g}(\vec{f}(\vec{x}))$

$$\frac{\partial \vec{g}}{\partial \vec{f}^T} = \begin{bmatrix} -\frac{f_2}{f_1^2} & \frac{1}{f_1} \\ f_2 & f_1 \end{bmatrix}$$

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{\partial \vec{f}}{\partial \vec{x}^T} = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\frac{\partial \vec{y}}{\partial \vec{x}^T} = \frac{\partial \vec{y}}{\partial \vec{f}^T}\,\frac{\partial f}{\partial \vec{x}^T} = \cdots\cdots$$

9.4. $d f(\vec{x}) = e^{-\frac{1}{2}\vec{x}^T A \vec{x}}\, d\left(-\frac{1}{2}\vec{x}^T A \vec{x}\right)$

$= \cdots \quad \cdots$

$\vdots$

$= - f(\vec{x}) . \vec{x}^T A\, d\vec{x}.$

$\Rightarrow \quad \frac{\partial}{\partial \vec{x}^T} f(\vec{x}) = \cdots$

9.5. $d f(\vec{x}) = \dfrac{(\vec{x}^T B \vec{x})\, d(\vec{x}^T A \vec{x}) - (\vec{x}^T A \vec{x})\, d(\vec{x}^T B \vec{x})}{(\vec{x}^T B \vec{x})^2}$

$= \cdots \quad \cdots$

9.6. $d(\vec{x}\vec{x}^T) = (d\vec{x})\vec{x}^T + \vec{x}(d\vec{x})^T$

So $d\,\mathrm{vec}(\vec{x}\vec{x}^T) = \mathrm{vec}((d\vec{x})\vec{x}^T) + \mathrm{vec}(\vec{x}(d\vec{x})^T)$

$\overset{\text{Thm 8.9 (b)}}{=} \vec{x} \otimes d\vec{x} + d\vec{x} \otimes \vec{x}.$

$\overset{\text{Thm 8.26 (c)}}{=} \vec{x} \otimes d\vec{x} + K_{mm}(\vec{x} \otimes d\vec{x})$

$\overset{\text{Ex 8.4 (b)}}{=} (\vec{x} \otimes I_m)\, d\vec{x} + K_{mm}(\vec{x} \otimes I_m)\, d\vec{x}.$

$= (I_{m^2} + K_{mm})(\vec{x} \otimes I_m)\, d\vec{x}.$

So $\frac{\partial f}{\partial \vec{x}^T} \operatorname{vec}(\vec{x}\vec{x}^T) = \cdots\cdots$

9.7 (a) $d\,\mathrm{tr}(AX) = \mathrm{tr}(A\,dX)$

$= \operatorname{vec}(A^T)^T \operatorname{vec}(dX)$

$= \operatorname{vec}(A^T)^T\, d\operatorname{vec}(X)$

So $\frac{\partial\, \mathrm{tr}(AX)}{\partial \operatorname{vec}(X)^T} = \cdots\cdots$

(b) $d\,\mathrm{tr}(AXBX) = \mathrm{tr}(A(dX)BX) + \mathrm{tr}(AXB\,dX)$

$= \mathrm{tr}(BXA(dX)) + \mathrm{tr}(AXB\,dX)$

$= \operatorname{vec}([BXA]^T + [AXB]^T)^T \operatorname{vec}(dX)$

$=$

$\vdots$

$\frac{\partial\, \mathrm{tr}(AXBX)}{\partial \operatorname{vec}(X)^T} = \cdots$

9.8. (a) $d|X^2| = d|X|^2 = 2|X|\,d|X| \overset{\text{Thm 9.1(c)}}{=} \cdots$ | $\frac{\partial |X|^2}{\partial \operatorname{vec}(X)^T} = \cdots$

(b) $d\,\mathrm{tr}(AX^{-1}) = \mathrm{tr}(A\,dX^{-1}) \overset{\text{Thm 9.2}}{=} \cdots$ | $\frac{\partial\, \mathrm{tr}(AX^{-1})}{\partial \operatorname{vec}(X)^T} = \cdots$

(c) $d\, \vec{a}^T X^{-1} \vec{a} = \vec{a}^T\, dX^{-1}\, \vec{a}$

$\overset{\text{Thm 9.2}}{=} \cdots$

use Thm 8.11, i.e.,

$\mathrm{Vec}(ABC) = (C^T \otimes A)\,\mathrm{Vec}(B)$

$$\frac{\partial\, \vec{a}^T X^{-1} \vec{a}}{\partial\, \mathrm{vec}(X)^T} = \cdots$$

9.9. By Corollary 9.1.1 on p. 397

$$d \log|X^T A X| = \mathrm{tr}\left((X^T A X)^{-1} d(X^T A X) \right)$$

$$\vdots$$
$$=$$
$$\vdots$$

$$\frac{\partial \log|X^T A X|}{\partial\, \mathrm{vec}(X)^T} = \cdots$$

9.10. By Thm 9.1 (c). (have $X^T X$ is nonsingular)

$$d\,|X^T X| = |X^T X|\, \mathrm{tr}\left((X^T X)^{-1} d(X^T X) \right)$$

$$=$$
$$\vdots$$

$$\frac{\partial\, |X^T X|}{\partial\, \mathrm{vec}(X)^T} = \cdots$$

9.12.

(a) $d\,\mathrm{vec}(AXB) = \mathrm{vec}(A(dX)B)$

$$\overset{\text{Thm 8.11}}{=} (B^T \otimes A)\,\mathrm{vec}(dX).$$

⋮

$$\frac{\partial\,\mathrm{vec}(AXB)}{\partial\,\mathrm{vec}(X)^T} = \cdots$$

(b) $d\,\mathrm{vec}(AX^{-1}B) = \mathrm{vec}(A\,dX^{-1}B) \overset{\text{Thm 9.2}}{=} \mathrm{vec}(\cdots\cdots).$

$$\frac{\partial\,\mathrm{vec}(AX^{-1}B)}{\partial\,\mathrm{vec}(X)^T} = \cdots$$