Exercise 3.3.30 Let $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ where B and C are square matrices.

a. Show that $c_A(x) = c_B(x)c_C(x)$.

b. If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of B and C, respectively, show that $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix}$ are eigenvectors of A, and show how every eigenvector of A arises from such eigenvectors.

Exercise 3.3.31 Referring to the model in Example 3.3.1, determine if the population stabilizes, becomes extinct, or becomes large in each case. Denote the adult and juvenile survival rates as A and J, and the reproduction rate as R.

	R	A	J
$a.$	2	$\frac{1}{2}$	$\frac{1}{2}$
$b.$	3	$\frac{1}{4}$	$\frac{1}{4}$
$c.$	2	$\frac{1}{4}$	$\frac{1}{3}$
$d.$	3	$\frac{3}{5}$	$\frac{1}{5}$

Exercise 3.3.32 In the model of Example 3.3.1, does the final outcome depend on the initial population of adult and juvenile females? Support your answer.

Exercise 3.3.33 In Example 3.3.1, keep the same reproduction rate of 2 and the same adult survival rate of $\frac{1}{2}$, but suppose that the juvenile survival rate is ρ. Determine which values of ρ cause the population to become extinct or to become large.

Exercise 3.3.34 In Example 3.3.1, let the juvenile survival rate be $\frac{2}{5}$ and let the reproduction rate be 2. What values of the adult survival rate α will ensure that the population stabilizes?

3.4 An Application to Linear Recurrences

It often happens that a problem can be solved by finding a sequence of numbers $x_0, x_1, x_2, \dots$ where the first few are known, and subsequent numbers are given in terms of earlier ones. Here is a combinatorial example where the object is to count the number of ways to do something.

Example 3.4.1

An urban planner wants to determine the number x_k of ways that a row of k parking spaces can be filled with cars and trucks if trucks take up two spaces each. Find the first few values of x_k.

Solution. Clearly, $x_0 = 1$ and $x_1 = 1$, while $x_2 = 2$ since there can be two cars or one truck. We have $x_3 = 3$ (the 3 configurations are *ccc*, *cT*, and *Tc*) and $x_4 = 5$ (*cccc, ccT, cTc, Tcc,* and *TT*). The key to this method is to find a way to express each subsequent x_k in terms of earlier values. In this case we claim that

$$x_{k+2} = x_k + x_{k+1} \text{ for every } k \geq 0 \tag{3.11}$$

Indeed, every way to fill $k+2$ spaces falls into one of two categories: Either a car is parked in the first space (and the remaining $k+1$ spaces are filled in x_{k+1} ways), or a truck is parked in the first two spaces (with the other k spaces filled in x_k ways). Hence, there are $x_{k+1} + x_k$ ways to fill the $k+2$ spaces. This is Equation 3.11.

The recurrence in Equation 3.11 determines x_k for every $k \geq 2$ since x_0 and x_1 are given. In fact, the first few values are

$$\begin{array}{rcrcr} x_0 & = & 1 & & \\ x_1 & = & 1 & & \\ x_2 & = & x_0 + x_1 & = & 2 \\ x_3 & = & x_1 + x_2 & = & 3 \\ x_4 & = & x_2 + x_3 & = & 5 \\ x_5 & = & x_3 + x_4 & = & 8 \\ \vdots & & \vdots & & \vdots \end{array}$$

Clearly, we can find x_k for any value of k, but one wishes for a "formula" for x_k as a function of k. It turns out that such a formula can be found using diagonalization. We will return to this example later.

A sequence $x_0, x_1, x_2, \dots$ of numbers is said to be given **recursively** if each number in the sequence is completely determined by those that come before it. Such sequences arise frequently in mathematics and computer science, and also occur in other parts of science. The formula $x_{k+2} = x_{k+1} + x_k$ in Example 3.4.1 is an example of a **linear recurrence relation** of length 2 because x_{k+2} is the sum of the two preceding terms x_{k+1} and x_k; in general, the **length** is m if x_{k+m} is a sum of multiples of $x_k, x_{k+1}, \dots, x_{k+m-1}$.

The simplest linear recursive sequences are of length 1, that is x_{k+1} is a fixed multiple of x_k for each k, say $x_{k+1} = ax_k$. If x_0 is specified, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and $x_3 = ax_2 = a^3x_0, \dots$. Continuing, we obtain $x_k = a^kx_0$ for each $k \geq 0$, which is an explicit formula for x_k as a function of k (when x_0 is given).

Such formulas are not always so easy to find for all choices of the initial values. Here is an example where diagonalization helps.

Example 3.4.2

Suppose the numbers $x_0, x_1, x_2, \dots$ are given by the linear recurrence relation

$$x_{k+2} = x_{k+1} + 6x_k \text{ for } k \geq 0$$

where x_0 and x_1 are specified. Find a formula for x_k when $x_0 = 1$ and $x_1 = 3$, and also when $x_0 = 1$ and $x_1 = 1$.

Solution. If $x_0 = 1$ and $x_1 = 3$, then

$$x_2 = x_1 + 6x_0 = 9, \quad x_3 = x_2 + 6x_1 = 27, \quad x_4 = x_3 + 6x_2 = 81$$

and it is apparent that

$$x_k = 3^k \text{ for } k = 0,\ 1,\ 2,\ 3, \text{ and } 4$$

This formula holds for all k because it is true for $k = 0$ and $k = 1$, and it satisfies the recurrence $x_{k+2} = x_{k+1} + 6x_k$ for each k as is readily checked.

However, if we begin instead with $x_0 = 1$ and $x_1 = 1$, the sequence continues

$$x_2 = 7, \quad x_3 = 13, \quad x_4 = 55, \quad x_5 = 133, \quad \dots$$

In this case, the sequence is uniquely determined but no formula is apparent. Nonetheless, a simple device transforms the recurrence into a matrix recurrence to which our diagonalization techniques apply.

The idea is to compute the sequence $\mathbf{v}_0,\ \mathbf{v}_1,\ \mathbf{v}_2,\ \dots$ of columns instead of the numbers $x_0,\ x_1,\ x_2,\ \dots,$ where

$$\mathbf{v}_k = \left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right] \text{ for each } k \geq 0$$

Then $\mathbf{v}_0 = \left[\begin{array}{c} x_0 \\ x_1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$ is specified, and the numerical recurrence $x_{k+2} = x_{k+1} + 6x_k$ transforms into a matrix recurrence as follows:

$$\mathbf{v}_{k+1} = \left[\begin{array}{c} x_{k+1} \\ x_{k+2} \end{array}\right] = \left[\begin{array}{c} x_{k+1} \\ 6x_k + x_{k+1} \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 6 & 1 \end{array}\right] \left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right] = A\mathbf{v}_k$$

where $A = \left[\begin{array}{cc} 0 & 1 \\ 6 & 1 \end{array}\right]$. Thus these columns $\mathbf{v}_k$ are a linear dynamical system, so Theorem 3.3.7 applies provided the matrix A is diagonalizable.

We have $c_A(x) = (x-3)(x+2)$ so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 3 \end{array}\right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} -1 \\ 2 \end{array}\right]$ as the reader can check. Since $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ 3 & 2 \end{array}\right]$ is invertible, it is a diagonalizing matrix for A. The coefficients b_i in Theorem 3.3.7 are given by $\left[\begin{array}{c} b_1 \\ b_2 \end{array}\right] = P^{-1}\mathbf{v}_0 = \left[\begin{array}{c} \frac{3}{5} \\ \frac{-2}{5} \end{array}\right]$, so that the theorem gives

$$\left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right] = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \tfrac{3}{5}3^k \left[\begin{array}{c} 1 \\ 3 \end{array}\right] + \tfrac{-2}{5}(-2)^k \left[\begin{array}{c} -1 \\ 2 \end{array}\right]$$

Equating top entries yields

$$x_k = \tfrac{1}{5}\left[3^{k+1} - (-2)^{k+1}\right] \text{ for } k \geq 0$$

This gives $x_0 = 1 = x_1$, and it satisfies the recurrence $x_{k+2} = x_{k+1} + 6x_k$ as is easily verified. Hence, it is the desired formula for the x_k.

Returning to Example 3.4.1, these methods give an exact formula and a good approximation for the numbers x_k in that problem.

Example 3.4.3

In Example 3.4.1, an urban planner wants to determine x_k, the number of ways that a row of k parking spaces can be filled with cars and trucks if trucks take up two spaces each. Find a formula for x_k and estimate it for large k.

Solution. We saw in Example 3.4.1 that the numbers x_k satisfy a linear recurrence

$$x_{k+2} = x_k + x_{k+1} \text{ for every } k \geq 0$$

If we write $\mathbf{v}_k = \left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right]$ as before, this recurrence becomes a matrix recurrence for the $\mathbf{v}_k$:

$$\mathbf{v}_{k+1} = \left[\begin{array}{c} x_{k+1} \\ x_{k+2} \end{array}\right] = \left[\begin{array}{c} x_{k+1} \\ x_k + x_{k+1} \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right] \left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right] = A\mathbf{v}_k$$

for all $k \geq 0$ where $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right]$. Moreover, A is diagonalizable here. The characteristic polynomial is $c_A(x) = x^2 - x - 1$ with roots $\frac{1}{2}\left[1 \pm \sqrt{5}\right]$ by the quadratic formula, so A has eigenvalues

$$\lambda_1 = \frac{1}{2}\left(1 + \sqrt{5}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(1 - \sqrt{5}\right)$$

Corresponding eigenvectors are $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ \lambda_1 \end{array}\right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ \lambda_2 \end{array}\right]$ respectively as the reader can verify. As the matrix $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ \lambda_1 & \lambda_2 \end{array}\right]$ is invertible, it is a diagonalizing matrix for A. We compute the coefficients b_1 and b_2 (in Theorem 3.3.7) as follows:

$$\left[\begin{array}{c} b_1 \\ b_2 \end{array}\right] = P^{-1}\mathbf{v}_0 = \frac{1}{-\sqrt{5}} \left[\begin{array}{cc} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \frac{1}{\sqrt{5}} \left[\begin{array}{c} \lambda_1 \\ -\lambda_2 \end{array}\right]$$

where we used the fact that $\lambda_1 + \lambda_2 = 1$. Thus Theorem 3.3.7 gives

$$\left[\begin{array}{c} x_k \\ x_{k+1} \end{array}\right] = \mathbf{v}_k = b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 = \frac{\lambda_1}{\sqrt{5}}\lambda_1^k \left[\begin{array}{c} 1 \\ \lambda_1 \end{array}\right] - \frac{\lambda_2}{\sqrt{5}}\lambda_2^k \left[\begin{array}{c} 1 \\ \lambda_2 \end{array}\right]$$

Comparing top entries gives an exact formula for the numbers x_k:

$$x_k = \frac{1}{\sqrt{5}}\left[\lambda_1^{k+1} - \lambda_2^{k+1}\right] \text{ for } k \geq 0$$

Finally, observe that λ_1 is dominant here (in fact, $\lambda_1 = 1.618$ and $\lambda_2 = -0.618$ to three decimal places) so λ_2^{k+1} is negligible compared with λ_1^{k+1} is large. Thus,

$$x_k \approx \frac{1}{\sqrt{5}}\lambda_1^{k+1} \text{ for each } k \geq 0.$$

This is a good approximation, even for as small a value as $k = 12$. Indeed, repeated use of the recurrence $x_{k+2} = x_k + x_{k+1}$ gives the exact value $x_{12} = 233$, while the approximation is $x_{12} \approx \frac{(1.618)^{13}}{\sqrt{5}} = 232.94$.

The sequence $x_0, x_1, x_2, \dots$ in Example 3.4.3 was first discussed in 1202 by Leonardo Pisano of Pisa, also known as Fibonacci,[15] and is now called the **Fibonacci sequence**. It is completely determined by the conditions $x_0 = 1$, $x_1 = 1$ and the recurrence $x_{k+2} = x_k + x_{k+1}$ for each $k \geq 0$. These numbers have

[15]Fibonacci was born in Italy. As a young man he travelled to India where he encountered the "Fibonacci" sequence. He returned to Italy and published this in his book *Liber Abaci* in 1202. In the book he is the first to bring the Hindu decimal system for representing numbers to Europe.

been studied for centuries and have many interesting properties (there is even a journal, the *Fibonacci Quarterly*, devoted exclusively to them). For example, biologists have discovered that the arrangement of leaves around the stems of some plants follow a Fibonacci pattern. The formula $x_k = \frac{1}{\sqrt{5}}\left[\lambda_1^{k+1} - \lambda_2^{k+1}\right]$ in Example 3.4.3 is called the **Binet formula**. It is remarkable in that the x_k are integers but λ_1 and λ_2 are not. This phenomenon can occur even if the eigenvalues λ_i are nonreal complex numbers.

We conclude with an example showing that *nonlinear* recurrences can be very complicated.

Example 3.4.4

Suppose a sequence $x_0, x_1, x_2, \dots$ satisfies the following recurrence:

$$x_{k+1} = \begin{cases} \frac{1}{2}x_k & \text{if } x_k \text{ is even} \\ 3x_k + 1 & \text{if } x_k \text{ is odd} \end{cases}$$

If $x_0 = 1$, the sequence is $1, 4, 2, 1, 4, 2, 1, \dots$ and so continues to cycle indefinitely. The same thing happens if $x_0 = 7$. Then the sequence is

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \dots$$

and it again cycles. However, it is not known whether every choice of x_0 will lead eventually to 1. It is quite possible that, for some x_0, the sequence will continue to produce different values indefinitely, or will repeat a value and cycle without reaching 1. No one knows for sure.

Exercises for 3.4

Exercise 3.4.1 Solve the following linear recurrences.

a. $x_{k+2} = 3x_k + 2x_{k+1}$, where $x_0 = 1$ and $x_1 = 1$.

b. $x_{k+2} = 2x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 2$.

c. $x_{k+2} = 2x_k + x_{k+1}$, where $x_0 = 0$ and $x_1 = 1$.

d. $x_{k+2} = 6x_k - x_{k+1}$, where $x_0 = 1$ and $x_1 = 1$.

Exercise 3.4.2 Solve the following linear recurrences.

a. $x_{k+3} = 6x_{k+2} - 11x_{k+1} + 6x_k$, where $x_0 = 1$, $x_1 = 0$, and $x_2 = 1$.

b. $x_{k+3} = -2x_{k+2} + x_{k+1} + 2x_k$, where $x_0 = 1$, $x_1 = 0$, and $x_2 = 1$.

[*Hint*: Use $\mathbf{v}_k = \left[\begin{array}{c} x_k \\ x_{k+1} \\ x_{k+2} \end{array}\right]$.]

Exercise 3.4.3 In Example 3.4.1 suppose buses are also allowed to park, and let x_k denote the number of ways a row of k parking spaces can be filled with cars, trucks, and buses.

a. If trucks and buses take up 2 and 3 spaces respectively, show that $x_{k+3} = x_k + x_{k+1} + x_{k+2}$ for each k, and use this recurrence to compute x_{10}. [*Hint*: The eigenvalues are of little use.]

b. If buses take up 4 spaces, find a recurrence for the x_k and compute x_{10}.

Exercise 3.4.4 A man must climb a flight of k steps. He always takes one or two steps at a time. Thus he can climb 3 steps in the following ways: 1, 1, 1; 1, 2; or 2, 1. Find s_k, the number of ways he can climb the flight of k steps. [*Hint*: Fibonacci.]

Exercise 3.4.5 How many "words" of k letters can be made from the letters $\{a, b\}$ if there are no adjacent a's?

Exercise 3.4.6 How many sequences of k flips of a coin are there with no HH?

Exercise 3.4.7 Find x_k, the number of ways to make a stack of k poker chips if only red, blue, and gold chips are used and no two gold chips are adjacent. [*Hint*: Show that $x_{k+2} = 2x_{k+1} + 2x_k$ by considering how many stacks have a red, blue, or gold chip on top.]

Exercise 3.4.8 A nuclear reactor contains α- and β-particles. In every second each α-particle splits into three β-particles, and each β-particle splits into an α-particle and two β-particles. If there is a single α-particle in the reactor at time $t = 0$, how many α-particles are there at $t = 20$ seconds? [*Hint*: Let x_k and y_k denote the number of α- and β-particles at time $t = k$ seconds. Find x_{k+1} and y_{k+1} in terms of x_k and y_k.]

Exercise 3.4.9 The annual yield of wheat in a certain country has been found to equal the average of the yield in the previous two years. If the yields in 1990 and 1991 were 10 and 12 million tons respectively, find a formula for the yield k years after 1990. What is the long-term average yield?

Exercise 3.4.10 Find the general solution to the recurrence $x_{k+1} = rx_k + c$ where r and c are constants. [*Hint*: Consider the cases $r = 1$ and $r \neq 1$ separately. If $r \neq 1$, you will need the identity $1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$ for $n \geq 1$.]

Exercise 3.4.11 Consider the length 3 recurrence $x_{k+3} = ax_k + bx_{k+1} + cx_{k+2}$.

a. If $\mathbf{v}_k = \begin{bmatrix} x_k \\ x_{k+1} \\ x_{k+2} \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ show that $\mathbf{v}_{k+1} = A\mathbf{v}_k$.

b. If λ is any eigenvalue of A, show that $\mathbf{x} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$ is a λ-eigenvector.

[*Hint*: Show directly that $A\mathbf{x} = \lambda\mathbf{x}$.]

c. Generalize (a) and (b) to a recurrence

$$x_{k+4} = ax_k + bx_{k+1} + cx_{k+2} + dx_{k+3}$$

of length 4.

Exercise 3.4.12 Consider the recurrence

$$x_{k+2} = ax_{k+1} + bx_k + c$$

where c may not be zero.

a. If $a + b \neq 1$ show that p can be found such that, if we set $y_k = x_k + p$, then $y_{k+2} = ay_{k+1} + by_k$. [Hence, the sequence x_k can be found provided y_k can be found by the methods of this section (or otherwise).]

b. Use (a) to solve $x_{k+2} = x_{k+1} + 6x_k + 5$ where $x_0 = 1$ and $x_1 = 1$.

Exercise 3.4.13 Consider the recurrence

$$x_{k+2} = ax_{k+1} + bx_k + c(k) \tag{3.12}$$

where $c(k)$ is a function of k, and consider the related recurrence

$$x_{k+2} = ax_{k+1} + bx_k \tag{3.13}$$

Suppose that $x_k = p_k$ is a particular solution of Equation 3.12.

a. If q_k is any solution of Equation 3.13, show that $q_k + p_k$ is a solution of Equation 3.12.

b. Show that every solution of Equation 3.12 arises as in (a) as the sum of a solution of Equation 3.13 plus the particular solution p_k of Equation 3.12.

3.5 An Application to Systems of Differential Equations

A function f of a real variable is said to be **differentiable** if its derivative exists and, in this case, we let f' denote the derivative. If f and g are differentiable functions, a system

$$\begin{aligned} f' &= 3f + 5g \\ g' &= -f + 2g \end{aligned}$$

is called a *system of first order differential equations,* or a *differential system* for short. Solving many practical problems often comes down to finding sets of functions that satisfy such a system (often involving more than two functions). In this section we show how diagonalization can help. Of course an acquaintance with calculus is required.

The Exponential Function

The simplest differential system is the following single equation:

$$f' = af \text{ where } a \text{ is constant} \tag{3.14}$$

It is easily verified that $f(x) = e^{ax}$ is one solution; in fact, Equation 3.14 is simple enough for us to find *all* solutions. Suppose that f is any solution, so that $f'(x) = af(x)$ for all x. Consider the new function g given by $g(x) = f(x)e^{-ax}$. Then the product rule of differentiation gives

$$\begin{aligned} g'(x) &= f(x)\left[-ae^{-ax}\right] + f'(x)e^{-ax} \\ &= -af(x)e^{-ax} + [af(x)]\,e^{-ax} \\ &= 0 \end{aligned}$$

for all x. Hence the function $g(x)$ has zero derivative and so must be a constant, say $g(x) = c$. Thus $c = g(x) = f(x)e^{-ax}$, that is

$$f(x) = ce^{ax}$$

In other words, every solution $f(x)$ of Equation 3.14 is just a scalar multiple of e^{ax}. Since every such scalar multiple is easily seen to be a solution of Equation 3.14, we have proved

Theorem 3.5.1

The set of solutions to $f' = af$ is $\{ce^{ax} \mid c \text{ any constant}\} = \mathbb{R}e^{ax}$.

Remarkably, this result together with diagonalization enables us to solve a wide variety of differential systems.