3.5 An Application to Systems of Differential Equations

A function f of a real variable is said to be **differentiable** if its derivative exists and, in this case, we let f' denote the derivative. If f and g are differentiable functions, a system

$$\begin{aligned} f' &= 3f + 5g \\ g' &= -f + 2g \end{aligned}$$

is called a *system of first order differential equations,* or a *differential system* for short. Solving many practical problems often comes down to finding sets of functions that satisfy such a system (often involving more than two functions). In this section we show how diagonalization can help. Of course an acquaintance with calculus is required.

The Exponential Function

The simplest differential system is the following single equation:

$$f' = af \text{ where } a \text{ is constant} \tag{3.14}$$

It is easily verified that $f(x) = e^{ax}$ is one solution; in fact, Equation 3.14 is simple enough for us to find *all* solutions. Suppose that f is any solution, so that $f'(x) = af(x)$ for all x. Consider the new function g given by $g(x) = f(x)e^{-ax}$. Then the product rule of differentiation gives

$$\begin{aligned} g'(x) &= f(x)\left[-ae^{-ax}\right] + f'(x)e^{-ax} \\ &= -af(x)e^{-ax} + [af(x)]\,e^{-ax} \\ &= 0 \end{aligned}$$

for all x. Hence the function $g(x)$ has zero derivative and so must be a constant, say $g(x) = c$. Thus $c = g(x) = f(x)e^{-ax}$, that is

$$f(x) = ce^{ax}$$

In other words, every solution $f(x)$ of Equation 3.14 is just a scalar multiple of e^{ax}. Since every such scalar multiple is easily seen to be a solution of Equation 3.14, we have proved

Theorem 3.5.1

The set of solutions to $f' = af$ is $\{ce^{ax} \mid c \text{ any constant}\} = \mathbb{R}e^{ax}$.

Remarkably, this result together with diagonalization enables us to solve a wide variety of differential systems.

Example 3.5.1

Assume that the number $n(t)$ of bacteria in a culture at time t has the property that the rate of change of n is proportional to n itself. If there are n_0 bacteria present when $t = 0$, find the number at time t.

Solution. Let k denote the proportionality constant. The rate of change of $n(t)$ is its time-derivative $n'(t)$, so the given relationship is $n'(t) = kn(t)$. Thus Theorem 3.5.1 shows that all solutions n are given by $n(t) = ce^{kt}$, where c is a constant. In this case, the constant c is determined by the requirement that there be n_0 bacteria present when $t = 0$. Hence $n_0 = n(0) = ce^{k0} = c$, so

$$n(t) = n_0 e^{kt}$$

gives the number at time t. Of course the constant k depends on the strain of bacteria.

The condition that $n(0) = n_0$ in Example 3.5.1 is called an **initial condition** or a **boundary condition** and serves to select one solution from the available solutions.

General Differential Systems

Solving a variety of problems, particularly in science and engineering, comes down to solving a system of linear differential equations. Diagonalization enters into this as follows. The general problem is to find differentiable functions $f_1, f_2, \dots, f_n$ that satisfy a system of equations of the form

$$\begin{array}{l} f_1' = a_{11}f_1 + a_{12}f_2 + \cdots + a_{1n}f_n \\ f_2' = a_{21}f_1 + a_{22}f_2 + \cdots + a_{2n}f_n \\ \vdots \qquad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ f_n' = a_{n1}f_1 + a_{n2}f_2 + \cdots + a_{nn}f_n \end{array}$$

where the a_{ij} are constants. This is called a **linear system of differential equations** or simply a **differential system**. The first step is to put it in matrix form. Write

$$\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \end{array} \right] \quad \mathbf{f}' = \left[\begin{array}{c} f_1' \\ f_2' \\ \vdots \\ f_n' \end{array} \right] \quad A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right]$$

Then the system can be written compactly using matrix multiplication:

$$\mathbf{f}' = A\mathbf{f}$$

Hence, given the matrix A, the problem is to find a column $\mathbf{f}$ of differentiable functions that satisfies this condition. This can be done if A is diagonalizable. Here is an example.

Example 3.5.2

Find a solution to the system

$$\begin{array}{rcl} f_1' &=& f_1 + 3f_2 \\ f_2' &=& 2f_1 + 2f_2 \end{array}$$

that satisfies $f_1(0) = 0,\ f_2(0) = 5$.

Solution. This is $\mathbf{f}' = A\mathbf{f}$, where $\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \end{array} \right]$ and $A = \left[\begin{array}{cc} 1 & 3 \\ 2 & 2 \end{array} \right]$. The reader can verify that $c_A(x) = (x-4)(x+1)$, and that $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 1 \end{array} \right]$ and $\mathbf{x}_2 = \left[\begin{array}{c} 3 \\ -2 \end{array} \right]$ are eigenvectors corresponding to the eigenvalues 4 and -1, respectively. Hence the diagonalization algorithm gives $P^{-1}AP = \left[\begin{array}{cc} 4 & 0 \\ 0 & -1 \end{array} \right]$, where $P = \left[\begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array} \right]$. Now consider new functions g_1 and g_2 given by $\mathbf{f} = P\mathbf{g}$ (equivalently, $\mathbf{g} = P^{-1}\mathbf{f}$), where $\mathbf{g} = \left[\begin{array}{c} g_1 \\ g_2 \end{array} \right]$ Then

$$\left[\begin{array}{c} f_1 \\ f_2 \end{array} \right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array} \right] \left[\begin{array}{c} g_1 \\ g_2 \end{array} \right] \quad \text{that is,} \quad \begin{array}{l} f_1 = g_1 + 3g_2 \\ f_2 = g_1 - 2g_2 \end{array}$$

Hence $f_1' = g_1' + 3g_2'$ and $f_2' = g_1' - 2g_2'$ so that

$$\mathbf{f}' = \left[\begin{array}{c} f_1' \\ f_2' \end{array} \right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array} \right] \left[\begin{array}{c} g_1' \\ g_2' \end{array} \right] = P\mathbf{g}'$$

If this is substituted in $\mathbf{f}' = A\mathbf{f}$, the result is $P\mathbf{g}' = AP\mathbf{g}$, whence

$$\mathbf{g}' = P^{-1}AP\mathbf{g}$$

But this means that

$$\left[\begin{array}{c} g_1' \\ g_2' \end{array} \right] = \left[\begin{array}{cc} 4 & 0 \\ 0 & -1 \end{array} \right] \left[\begin{array}{c} g_1 \\ g_2 \end{array} \right], \quad \text{so} \quad \begin{array}{l} g_1' = 4g_1 \\ g_2' = -g_2 \end{array}$$

Hence Theorem 3.5.1 gives $g_1(x) = ce^{4x}$, $g_2(x) = de^{-x}$, where c and d are constants. Finally, then,

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array} \right] = P \left[\begin{array}{c} g_1(x) \\ g_2(x) \end{array} \right] = \left[\begin{array}{cc} 1 & 3 \\ 1 & -2 \end{array} \right] \left[\begin{array}{c} ce^{4x} \\ de^{-x} \end{array} \right] = \left[\begin{array}{c} ce^{4x} + 3de^{-x} \\ ce^{4x} - 2de^{-x} \end{array} \right]$$

so the *general solution* is

$$\begin{array}{rcl} f_1(x) &=& ce^{4x} + 3de^{-x} \\ f_2(x) &=& ce^{4x} - 2de^{-x} \end{array} \quad c \text{ and } d \text{ constants}$$

It is worth observing that this can be written in matrix form as

$$\left[\begin{array}{c} f_1(x) \\ f_2(x) \end{array} \right] = c \left[\begin{array}{c} 1 \\ 1 \end{array} \right] e^{4x} + d \left[\begin{array}{c} 3 \\ -2 \end{array} \right] e^{-x}$$

That is,

$$\mathbf{f}(x) = c\mathbf{x}_1 e^{4x} + d\mathbf{x}_2 e^{-x}$$

This form of the solution works more generally, as will be shown.
Finally, the requirement that $f_1(0) = 0$ and $f_2(0) = 5$ in this example determines the constants c and d:

$$\begin{aligned} 0 &= f_1(0) = ce^0 + 3de^0 = c + 3d \\ 5 &= f_2(0) = ce^0 - 2de^0 = c - 2d \end{aligned}$$

These equations give $c = 3$ and $d = -1$, so

$$\begin{aligned} f_1(x) &= 3e^{4x} - 3e^{-x} \\ f_2(x) &= 3e^{4x} + 2e^{-x} \end{aligned}$$

satisfy all the requirements.

The technique in this example works in general.

Theorem 3.5.2

Consider a linear system

$$\mathbf{f}' = A\mathbf{f}$$

of differential equations, where A is an $n \times n$ diagonalizable matrix. Let $P^{-1}AP$ be diagonal, where P is given in terms of its columns

$$P = [\mathbf{x}_1,\ \mathbf{x}_2,\ \cdots,\ \mathbf{x}_n]$$

and $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \dots,\ \mathbf{x}_n\}$ are eigenvectors of A. If $\mathbf{x}_i$ corresponds to the eigenvalue λ_i for each i, then every solution $\mathbf{f}$ of $\mathbf{f}' = A\mathbf{f}$ has the form

$$\mathbf{f}(x) = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

where $c_1,\ c_2,\ \dots,\ c_n$ are arbitrary constants.

Proof. By Theorem 3.3.4, the matrix $P = \left[\begin{array}{cccc} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{array}\right]$ is invertible and

$$P^{-1}AP = \left[\begin{array}{cccc} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{array}\right]$$

As in Example 3.5.2, write $\mathbf{f} = \left[\begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \end{array}\right]$ and define $\mathbf{g} = \left[\begin{array}{c} g_1 \\ g_2 \\ \vdots \\ g_n \end{array}\right]$ by $\mathbf{g} = P^{-1}\mathbf{f}$; equivalently, $\mathbf{f} = P\mathbf{g}$. If $P = \left[p_{ij}\right]$, this gives

$$f_i = p_{i1}g_1 + p_{i2}g_2 + \cdots + p_{in}g_n$$

Since the p_{ij} are constants, differentiation preserves this relationship:

$$f_i' = p_{i1}g_1' + p_{i2}g_2' + \cdots + p_{in}g_n'$$

so $\mathbf{f}' = P\mathbf{g}'$. Substituting this into $\mathbf{f}' = A\mathbf{f}$ gives $P\mathbf{g}' = AP\mathbf{g}$. But then left multiplication by P^{-1} gives $\mathbf{g}' = P^{-1}AP\mathbf{g}$, so the original system of equations $\mathbf{f}' = A\mathbf{f}$ for $\mathbf{f}$ becomes much simpler in terms of $\mathbf{g}$:

$$\begin{bmatrix} g_1' \\ g_2' \\ \vdots \\ g_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

Hence $g_i' = \lambda_i g_i$ holds for each i, and Theorem 3.5.1 implies that the only solutions are

$$g_i(x) = c_i e^{\lambda_i x} \quad c_i \text{ some constant}$$

Then the relationship $\mathbf{f} = P\mathbf{g}$ gives the functions $f_1, f_2, \dots, f_n$ as follows:

$$\mathbf{f}(x) = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] \begin{bmatrix} c_1 e^{\lambda_1 x} \\ c_2 e^{\lambda_2 x} \\ \vdots \\ c_n e^{\lambda_n x} \end{bmatrix} = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

This is what we wanted. □

The theorem shows that *every* solution to $\mathbf{f}' = A\mathbf{f}$ is a linear combination

$$\mathbf{f}(x) = c_1\mathbf{x}_1 e^{\lambda_1 x} + c_2\mathbf{x}_2 e^{\lambda_2 x} + \cdots + c_n\mathbf{x}_n e^{\lambda_n x}$$

where the coefficients c_i are arbitrary. Hence this is called the **general solution** to the system of differential equations. In most cases the solution functions $f_i(x)$ are required to satisfy boundary conditions, often of the form $f_i(a) = b_i$, where $a, b_1, \dots, b_n$ are prescribed numbers. These conditions determine the constants c_i. The following example illustrates this and displays a situation where one eigenvalue has multiplicity greater than 1.

Example 3.5.3

Find the general solution to the system

$$\begin{aligned} f_1' &= 5f_1 + 8f_2 + 16f_3 \\ f_2' &= 4f_1 + f_2 + 8f_3 \\ f_3' &= -4f_1 - 4f_2 - 11f_3 \end{aligned}$$

Then find a solution satisfying the boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$.

Solution. The system has the form $\mathbf{f}' = A\mathbf{f}$, where $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$. In this case $c_A(x) = (x+3)^2(x-1)$ and eigenvectors corresponding to the eigenvalues -3, -3, and 1 are, respectively,

$$\mathbf{x}_1 = \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array} \right] \quad \mathbf{x}_2 = \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array} \right] \quad \mathbf{x}_3 = \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array} \right]$$

Hence, by Theorem 3.5.2, the general solution is

$$\mathbf{f}(x) = c_1 \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array} \right] e^{-3x} + c_2 \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array} \right] e^{-3x} + c_3 \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array} \right] e^{x}, \quad c_i \text{ constants.}$$

The boundary conditions $f_1(0) = f_2(0) = f_3(0) = 1$ determine the constants c_i.

$$\begin{aligned} \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right] = \mathbf{f}(0) &= c_1 \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array} \right] + c_2 \left[\begin{array}{r} -2 \\ 0 \\ 1 \end{array} \right] + c_3 \left[\begin{array}{r} 2 \\ 1 \\ -1 \end{array} \right] \\ &= \left[\begin{array}{rrr} -1 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{array} \right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right] \end{aligned}$$

The solution is $c_1 = -3$, $c_2 = 5$, $c_3 = 4$, so the required specific solution is

$$\begin{aligned} f_1(x) &= -7e^{-3x} + 8e^{x} \\ f_2(x) &= -3e^{-3x} + 4e^{x} \\ f_3(x) &= 5e^{-3x} - 4e^{x} \end{aligned}$$

Exercises for 3.5

Exercise 3.5.1 Use Theorem 3.5.1 to find the general solution to each of the following systems. Then find a specific solution satisfying the given boundary condition.

a. $\begin{aligned} f_1' &= 2f_1 + 4f_2, \ f_1(0) = 0 \\ f_2' &= 3f_1 + 3f_2, \ f_2(0) = 1 \end{aligned}$

b. $\begin{aligned} f_1' &= -f_1 + 5f_2, \ f_1(0) = 1 \\ f_2' &= f_1 + 3f_2, \ f_2(0) = -1 \end{aligned}$

c. $\begin{aligned} f_1' &= 4f_2 + 4f_3 \\ f_2' &= f_1 + f_2 - 2f_3 \\ f_3' &= -f_1 + f_2 + 4f_3 \end{aligned}$

$f_1(0) = f_2(0) = f_3(0) = 1$

d. $\begin{aligned} f_1' &= 2f_1 + f_2 + 2f_3 \\ f_2' &= 2f_1 + 2f_2 - 2f_3 \\ f_3' &= 3f_1 + f_2 + f_3 \end{aligned}$

$f_1(0) = f_2(0) = f_3(0) = 1$

Exercise 3.5.2 Show that the solution to $f' = af$ satisfying $f(x_0) = k$ is $f(x) = ke^{a(x-x_0)}$.

Exercise 3.5.3 A radioactive element decays at a rate proportional to the amount present. Suppose an initial mass of 10 g decays to 8 g in 3 hours.

a. Find the mass t hours later.

b. Find the half-life of the element—the time taken to decay to half its mass.

Exercise 3.5.4 The population $N(t)$ of a region at time t increases at a rate proportional to the population. If the population doubles every 5 years and is 3 million initially, find $N(t)$.

Exercise 3.5.5 Let A be an invertible diagonalizable $n \times n$ matrix and let $\mathbf{b}$ be an n-column of constant functions. We can solve the system $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ as follows:

a. If $\mathbf{g}$ satisfies $\mathbf{g}' = A\mathbf{g}$ (using Theorem 3.5.2), show that $\mathbf{f} = \mathbf{g} - A^{-1}\mathbf{b}$ is a solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$.

b. Show that every solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ arises as in (a) for some solution $\mathbf{g}$ to $\mathbf{g}' = A\mathbf{g}$.

Exercise 3.5.6 Denote the second derivative of f by $f'' = (f')'$. Consider the second order differential equation

$$f'' - a_1 f' - a_2 f = 0, \quad a_1 \text{ and } a_2 \text{ real numbers} \quad (3.15)$$

a. If f is a solution to Equation 3.15 let $f_1 = f$ and $f_2 = f' - a_1 f$. Show that

$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 \end{cases},$$

that is $\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

b. Conversely, if $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a solution to the system in (a), show that f_1 is a solution to Equation 3.15.

Exercise 3.5.7 Writing $f''' = (f'')'$, consider the third order differential equation

$$f''' - a_1 f'' - a_2 f' - a_3 f = 0$$

where a_1, a_2, and a_3 are real numbers. Let $f_1 = f$, $f_2 = f' - a_1 f$ and $f_3 = f'' - a_1 f' - a_2 f''$.

a. Show that $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is a solution to the system

$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 + f_3, \\ f_3' = a_3 f_1 \end{cases}$$

that is $\begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

b. Show further that if $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is any solution to this system, then $f = f_1$ is a solution to Equation 3.15.

Remark. A similar construction casts every linear differential equation of order n (with constant coefficients) as an $n \times n$ linear system of first order equations. However, the matrix need not be diagonalizable, so other methods have been developed.

3.6 Proof of the Cofactor Expansion Theorem

Recall that our definition of the term *determinant* is inductive: The determinant of any 1×1 matrix is defined first; then it is used to define the determinants of 2×2 matrices. Then that is used for the 3×3 case, and so on. The case of a 1×1 matrix $[a]$ poses no problem. We simply define

$$\det [a] = a$$

as in Section 3.1. Given an $n \times n$ matrix A, define A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j. Now assume that the determinant of any $(n-1) \times (n-1)$ matrix has been defined. Then the determinant of A is *defined* to be

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{n+1} a_{n1} \det A_{n1} \\ &= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1} \end{aligned}$$