

Exercise 3.5.4 The population $N(t)$ of a region at time t increases at a rate proportional to the population. If the population doubles every 5 years and is 3 million initially, find $N(t)$.

Exercise 3.5.5 Let A be an invertible diagonalizable $n \times n$ matrix and let \mathbf{b} be an n -column of constant functions. We can solve the system $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ as follows:

- If \mathbf{g} satisfies $\mathbf{g}' = A\mathbf{g}$ (using Theorem 3.5.2), show that $\mathbf{f} = \mathbf{g} - A^{-1}\mathbf{b}$ is a solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$.
- Show that every solution to $\mathbf{f}' = A\mathbf{f} + \mathbf{b}$ arises as in (a) for some solution \mathbf{g} to $\mathbf{g}' = A\mathbf{g}$.

Exercise 3.5.6 Denote the second derivative of f by $f'' = (f')'$. Consider the second order differential equation

$$f'' - a_1f' - a_2f = 0, \quad a_1 \text{ and } a_2 \text{ real numbers} \quad (3.15)$$

- If f is a solution to Equation 3.15 let $f_1 = f$ and $f_2 = f' - a_1f$. Show that

$$\begin{cases} f_1' = a_1f_1 + f_2 \\ f_2' = a_2f_1 \end{cases},$$

$$\text{that is } \begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

- Conversely, if $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a solution to the system in (a), show that f_1 is a solution to Equation 3.15.

Exercise 3.5.7 Writing $f''' = (f'')'$, consider the third order differential equation

$$f''' - a_1f'' - a_2f' - a_3f = 0$$

where $a_1, a_2,$ and a_3 are real numbers. Let $f_1 = f, f_2 = f' - a_1f$ and $f_3 = f'' - a_1f' - a_2f$.

- Show that $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is a solution to the system

$$\begin{cases} f_1' = a_1f_1 + f_2 \\ f_2' = a_2f_1 + f_3 \\ f_3' = a_3f_1 \end{cases},$$

$$\text{that is } \begin{bmatrix} f_1' \\ f_2' \\ f_3' \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

- Show further that if $\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ is any solution to this system, then $f = f_1$ is a solution to Equation 3.15.

Remark. A similar construction casts every linear differential equation of order n (with constant coefficients) as an $n \times n$ linear system of first order equations. However, the matrix need not be diagonalizable, so other methods have been developed.

3.6 Proof of the Cofactor Expansion Theorem

Recall that our definition of the term *determinant* is inductive: The determinant of any 1×1 matrix is defined first; then it is used to define the determinants of 2×2 matrices. Then that is used for the 3×3 case, and so on. The case of a 1×1 matrix $[a]$ poses no problem. We simply define

$$\det [a] = a$$

as in Section 3.1. Given an $n \times n$ matrix A , define A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j . Now assume that the determinant of any $(n-1) \times (n-1)$ matrix has been defined. Then the determinant of A is *defined* to be

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{21} \det A_{21} + \cdots + (-1)^{n+1} a_{n1} \det A_{n1} \\ &= \sum_{i=1}^n (-1)^{i+1} a_{i1} \det A_{i1} \end{aligned}$$

where summation notation has been introduced for convenience.¹⁶ Observe that, in the terminology of Section 3.1, this is just the cofactor expansion of $\det A$ along the first column, and that $(-1)^{i+j} \det A_{ij}$ is the (i, j) -cofactor (previously denoted as $c_{ij}(A)$).¹⁷ To illustrate the definition, consider the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then the definition gives

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \det [a_{22}] - a_{21} \det [a_{12}] = a_{11}a_{22} - a_{21}a_{12}$$

and this is the same as the definition in Section 3.1.

Of course, the task now is to use this definition to *prove* that the cofactor expansion along *any* row or column yields $\det A$ (this is Theorem 3.1.1). The proof proceeds by first establishing the properties of determinants stated in Theorem 3.1.2 but for *rows* only (see Lemma 3.6.2). This being done, the full proof of Theorem 3.1.1 is not difficult. The proof of Lemma 3.6.2 requires the following preliminary result.

Lemma 3.6.1

Let $A, B,$ and C be $n \times n$ matrices that are identical except that the p th row of A is the sum of the p th rows of B and C . Then

$$\det A = \det B + \det C$$

Proof. We proceed by induction on n , the cases $n = 1$ and $n = 2$ being easily checked. Consider a_{i1} and A_{i1} :

Case 1: If $i \neq p$,

$$a_{i1} = b_{i1} = c_{i1} \quad \text{and} \quad \det A_{i1} = \det B_{i1} = \det C_{i1}$$

by induction because A_{i1}, B_{i1}, C_{i1} are identical except that one row of A_{i1} is the sum of the corresponding rows of B_{i1} and C_{i1} .

Case 2: If $i = p$,

$$a_{p1} = b_{p1} + c_{p1} \quad \text{and} \quad A_{p1} = B_{p1} = C_{p1}$$

Now write out the defining sum for $\det A$, splitting off the p th term for special attention.

$$\begin{aligned} \det A &= \sum_{i \neq p} a_{i1} (-1)^{i+1} \det A_{i1} + a_{p1} (-1)^{p+1} \det A_{p1} \\ &= \sum_{i \neq p} a_{i1} (-1)^{i+1} [\det B_{i1} + \det C_{i1}] + (b_{p1} + c_{p1}) (-1)^{p+1} \det A_{p1} \end{aligned}$$

where $\det A_{i1} = \det B_{i1} + \det C_{i1}$ by induction. But the terms here involving B_{i1} and b_{p1} add up to $\det B$ because $a_{i1} = b_{i1}$ if $i \neq p$ and $A_{p1} = B_{p1}$. Similarly, the terms involving C_{i1} and c_{p1} add up to $\det C$. Hence $\det A = \det B + \det C$, as required. \square

¹⁶Summation notation is a convenient shorthand way to write sums of similar expressions. For example $a_1 + a_2 + a_3 + a_4 = \sum_{i=1}^4 a_i$, $a_5b_5 + a_6b_6 + a_7b_7 + a_8b_8 = \sum_{k=5}^8 a_k b_k$, and $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \sum_{j=1}^5 j^2$.

¹⁷Note that we used the expansion along *row* 1 at the beginning of Section 3.1. The column 1 expansion definition is more convenient here.

Lemma 3.6.2

Let $A = [a_{ij}]$ denote an $n \times n$ matrix.

1. If $B = [b_{ij}]$ is formed from A by multiplying a row of A by a number u , then $\det B = u \det A$.
2. If A contains a row of zeros, then $\det A = 0$.
3. If $B = [b_{ij}]$ is formed by interchanging two rows of A , then $\det B = -\det A$.
4. If A contains two identical rows, then $\det A = 0$.
5. If $B = [b_{ij}]$ is formed by adding a multiple of one row of A to a different row, then $\det B = \det A$.

Proof. For later reference the defining sums for $\det A$ and $\det B$ are as follows:

$$\det A = \sum_{i=1}^n a_{i1} (-1)^{i+1} \det A_{i1} \quad (3.16)$$

$$\det B = \sum_{i=1}^n b_{i1} (-1)^{i+1} \det B_{i1} \quad (3.17)$$

Property 1. The proof is by induction on n , the cases $n = 1$ and $n = 2$ being easily verified. Consider the i th term in the sum 3.17 for $\det B$ where B is the result of multiplying row p of A by u .

- a. If $i \neq p$, then $b_{i1} = a_{i1}$ and $\det B_{i1} = u \det A_{i1}$ by induction because B_{i1} comes from A_{i1} by multiplying a row by u .
- b. If $i = p$, then $b_{p1} = ua_{p1}$ and $B_{p1} = A_{p1}$.

In either case, each term in Equation 3.17 is u times the corresponding term in Equation 3.16, so it is clear that $\det B = u \det A$.

Property 2. This is clear by property 1 because the row of zeros has a common factor $u = 0$.

Property 3. Observe first that it suffices to prove property 3 for interchanges of adjacent rows. (Rows p and q ($q > p$) can be interchanged by carrying out $2(q - p) - 1$ adjacent changes, which results in an *odd* number of sign changes in the determinant.) So suppose that rows p and $p + 1$ of A are interchanged to obtain B . Again consider the i th term in Equation 3.17.

- a. If $i \neq p$ and $i \neq p + 1$, then $b_{i1} = a_{i1}$ and $\det B_{i1} = -\det A_{i1}$ by induction because B_{i1} results from interchanging adjacent rows in A_{i1} . Hence the i th term in Equation 3.17 is the negative of the i th term in Equation 3.16. Hence $\det B = -\det A$ in this case.
- b. If $i = p$ or $i = p + 1$, then $b_{p1} = a_{p+1, 1}$ and $B_{p1} = A_{p+1, 1}$, whereas $b_{p+1, 1} = a_{p1}$ and $B_{p+1, 1} = A_{p1}$. Hence terms p and $p + 1$ in Equation 3.17 are

$$b_{p1} (-1)^{p+1} \det B_{p1} = -a_{p+1, 1} (-1)^{(p+1)+1} \det (A_{p+1, 1})$$

$$b_{p+1, 1} (-1)^{(p+1)+1} \det B_{p+1, 1} = -a_{p1} (-1)^{p+1} \det (A_{p1})$$

This means that terms p and $p + 1$ in Equation 3.17 are the same as these terms in Equation 3.16, except that the order is reversed and the signs are changed. Thus the sum 3.17 is the negative of the sum 3.16; that is, $\det B = -\det A$.

Property 4. If rows p and q in A are identical, let B be obtained from A by interchanging these rows. Then $B = A$ so $\det A = \det B$. But $\det B = -\det A$ by property 3 so $\det A = -\det A$. This implies that $\det A = 0$.

Property 5. Suppose B results from adding u times row q of A to row p . Then Lemma 3.6.1 applies to B to show that $\det B = \det A + \det C$, where C is obtained from A by replacing row p by u times row q . It now follows from properties 1 and 4 that $\det C = 0$ so $\det B = \det A$, as asserted. \square

These facts are enough to enable us to prove Theorem 3.1.1. For convenience, it is restated here in the notation of the foregoing lemmas. The only difference between the notations is that the (i, j) -cofactor of an $n \times n$ matrix A was denoted earlier by

$$c_{ij}(A) = (-1)^{i+j} \det A_{ij}$$

Theorem 3.6.1

If $A = [a_{ij}]$ is an $n \times n$ matrix, then

1. $\det A = \sum_{i=1}^n a_{ij}(-1)^{i+j} \det A_{ij}$ (cofactor expansion along column j).
2. $\det A = \sum_{j=1}^n a_{ij}(-1)^{i+j} \det A_{ij}$ (cofactor expansion along row i).

Here A_{ij} denotes the matrix obtained from A by deleting row i and column j .

Proof. Lemma 3.6.2 establishes the truth of Theorem 3.1.2 for rows. With this information, the arguments in Section 3.2 proceed exactly as written to establish that $\det A = \det A^T$ holds for any $n \times n$ matrix A . Now suppose B is obtained from A by interchanging two columns. Then B^T is obtained from A^T by interchanging two rows so, by property 3 of Lemma 3.6.2,

$$\det B = \det B^T = -\det A^T = -\det A$$

Hence property 3 of Lemma 3.6.2 holds for columns too.

This enables us to prove the cofactor expansion for columns. Given an $n \times n$ matrix $A = [a_{ij}]$, let $B = [b_{ij}]$ be obtained by moving column j to the left side, using $j - 1$ interchanges of adjacent columns. Then $\det B = (-1)^{j-1} \det A$ and, because $B_{i1} = A_{ij}$ and $b_{i1} = a_{ij}$ for all i , we obtain

$$\begin{aligned} \det A &= (-1)^{j-1} \det B = (-1)^{j-1} \sum_{i=1}^n b_{i1}(-1)^{i+1} \det B_{i1} \\ &= \sum_{i=1}^n a_{ij}(-1)^{i+j} \det A_{ij} \end{aligned}$$

This is the cofactor expansion of $\det A$ along column j .

Finally, to prove the row expansion, write $B = A^T$. Then $B_{ij} = (A^T_{ij})$ and $b_{ij} = a_{ji}$ for all i and j . Expanding $\det B$ along column j gives

$$\begin{aligned}\det A &= \det A^T = \det B = \sum_{i=1}^n b_{ij}(-1)^{i+j} \det B_{ij} \\ &= \sum_{i=1}^n a_{ji}(-1)^{j+i} \det [(A^T_{ji})] = \sum_{i=1}^n a_{ji}(-1)^{j+i} \det A_{ji}\end{aligned}$$

This is the required expansion of $\det A$ along row j . □

Exercises for 3.6

Exercise 3.6.1 Prove Lemma 3.6.1 for columns.

Exercise 3.6.2 Verify that interchanging rows p and q ($q > p$) can be accomplished using $2(q-p) - 1$ adjacent interchanges.

Exercise 3.6.3 If u is a number and A is an $n \times n$ matrix, prove that $\det(uA) = u^n \det A$ by induction on n , using only the definition of $\det A$.

Supplementary Exercises for Chapter 3

Exercise 3.1 Show that

$$\det \begin{bmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{bmatrix} = (1+x^3) \det \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$$

Exercise 3.2

- Show that $(A_{ij})^T = (A^T)_{ji}$ for all i, j , and all square matrices A .
- Use (a) to prove that $\det A^T = \det A$. [*Hint*: Induction on n where A is $n \times n$.]

Exercise 3.3 Show that $\det \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix} = (-1)^{nm}$ for all $n \geq 1$ and $m \geq 1$.

Exercise 3.4 Show that

$$\det \begin{bmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{bmatrix} = (b-a)(c-a)(c-b)(a+b+c)$$

Exercise 3.5 Let $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ be a 2×2 matrix with rows R_1 and R_2 . If $\det A = 5$, find $\det B$ where

$$B = \begin{bmatrix} 3R_1 + 2R_2 \\ 2R_1 + 5R_2 \end{bmatrix}$$

Exercise 3.6 Let $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ and let $\mathbf{v}_k = A^k \mathbf{v}_0$ for each $k \geq 0$.

a. Show that A has no dominant eigenvalue.

b. Find \mathbf{v}_k if \mathbf{v}_0 equals:

i. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ii. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

iii. $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$