Finally, to prove the row expansion, write $B = A^T$. Then $B_{ij} = (A^T_{ij})$ and $b_{ij} = a_{ji}$ for all i and j. Expanding $\det B$ along column j gives

$$\begin{aligned}\det A = \det A^T = \det B &= \sum_{i=1}^{n} b_{ij}(-1)^{i+j} \det B_{ij} \\ &= \sum_{i=1}^{n} a_{ji}(-1)^{j+i} \det \left[(A^T_{ji})\right] = \sum_{i=1}^{n} a_{ji}(-1)^{j+i} \det A_{ji}\end{aligned}$$

This is the required expansion of $\det A$ along row j. □

Exercises for 3.6

Exercise 3.6.1 Prove Lemma 3.6.1 for columns.

Exercise 3.6.2 Verify that interchanging rows p and q $(q > p)$ can be accomplished using $2(q-p)-1$ adjacent interchanges.

Exercise 3.6.3 If u is a number and A is an $n \times n$ matrix, prove that $\det(uA) = u^n \det A$ by induction on n, using only the definition of $\det A$.

Supplementary Exercises for Chapter 3

Exercise 3.1 Show that

$$\det \begin{bmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{bmatrix} = (1+x^3) \det \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$$

Exercise 3.2

a. Show that $(A_{ij})^T = (A^T)_{ji}$ for all i, j, and all square matrices A.

b. Use (a) to prove that $\det A^T = \det A$. [*Hint*: Induction on n where A is $n \times n$.]

Exercise 3.3 Show that $\det \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix} = (-1)^{nm}$ for all $n \geq 1$ and $m \geq 1$.

Exercise 3.4 Show that

$$\det \begin{bmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{bmatrix} = (b-a)(c-a)(c-b)(a+b+c)$$

Exercise 3.5 Let $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ be a 2×2 matrix with rows R_1 and R_2. If $\det A = 5$, find $\det B$ where

$$B = \begin{bmatrix} 3R_1 + 2R_3 \\ 2R_1 + 5R_2 \end{bmatrix}$$

Exercise 3.6 Let $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ and let $\mathbf{v}_k = A^k \mathbf{v}_0$ for each $k \geq 0$.

a. Show that A has no dominant eigenvalue.

b. Find $\mathbf{v}_k$ if $\mathbf{v}_0$ equals:

i. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ii. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

iii. $\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$