

- b. Find the rotation of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{6}$.

Exercise 4.4.5 Find the matrix of the rotation in \mathbb{R}^3 about the x axis through the angle θ (from the positive y axis to the positive z axis).

Exercise 4.4.6 Find the matrix of the rotation about the y axis through the angle θ (from the positive x axis to the positive z axis).

Exercise 4.4.7 If A is 3×3 , show that the image of the line in \mathbb{R}^3 through \mathbf{p}_0 with direction vector \mathbf{d} is the line through $A\mathbf{p}_0$ with direction vector $A\mathbf{d}$, assuming that $A\mathbf{d} \neq \mathbf{0}$. What happens if $A\mathbf{d} = \mathbf{0}$?

Exercise 4.4.8 If A is 3×3 and invertible, show that the image of the plane through the origin with normal \mathbf{n} is the plane through the origin with normal $\mathbf{n}_1 = B\mathbf{n}$ where $B = (A^{-1})^T$. [Hint: Use the fact that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$ to show that $\mathbf{n}_1 \cdot (A\mathbf{p}) = \mathbf{n} \cdot \mathbf{p}$ for each \mathbf{p} in \mathbb{R}^3 .]

Exercise 4.4.9 Let L be the line through the origin in \mathbb{R}^2 with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \mathbf{0}$.

- a. If P_L denotes projection on L , show that P_L has matrix $\frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.
- b. If Q_L denotes reflection in L , show that Q_L has matrix $\frac{1}{a^2+b^2} \begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$.

Exercise 4.4.10 Let \mathbf{n} be a nonzero vector in \mathbb{R}^3 , let L be the line through the origin with direction vector \mathbf{n} , and let M be the plane through the origin with normal \mathbf{n} . Show that $P_L(\mathbf{v}) = Q_L(\mathbf{v}) + P_M(\mathbf{v})$ for all \mathbf{v} in \mathbb{R}^3 . [In this case, we say that $P_L = Q_L + P_M$.]

Exercise 4.4.11 If M is the plane through the origin in \mathbb{R}^3 with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that Q_M has matrix

$$\frac{1}{a^2+b^2+c^2} \begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}$$

4.5 An Application to Computer Graphics

Computer graphics deals with images displayed on a computer screen, and so arises in a variety of applications, ranging from word processors, to *Star Wars* animations, to video games, to wire-frame images of an airplane. These images consist of a number of points on the screen, together with instructions on how to fill in areas bounded by lines and curves. Often curves are approximated by a set of short straight-line segments, so that the curve is specified by a series of points on the screen at the end of these segments. Matrix transformations are important here because matrix images of straight line segments are again line segments.¹⁴ Note that a colour image requires that three images are sent, one to each of the red, green, and blue phosphorus dots on the screen, in varying intensities.

Consider displaying the letter *A*. In reality, it is depicted on the screen, as in Figure 4.5.1, by specifying the coordinates of the 11 corners and filling in the interior.

For simplicity, we will disregard the thickness of the letter, so we require only five coordinates as in Figure 4.5.2.

¹⁴If \mathbf{v}_0 and \mathbf{v}_1 are vectors, the vector from \mathbf{v}_0 to \mathbf{v}_1 is $\mathbf{d} = \mathbf{v}_1 - \mathbf{v}_0$. So a vector \mathbf{v} lies on the line segment between \mathbf{v}_0 and \mathbf{v}_1 if and only if $\mathbf{v} = \mathbf{v}_0 + t\mathbf{d}$ for some number t in the range $0 \leq t \leq 1$. Thus the image of this segment is the set of vectors $A\mathbf{v} = A\mathbf{v}_0 + tA\mathbf{d}$ with $0 \leq t \leq 1$, that is the image is the segment between $A\mathbf{v}_0$ and $A\mathbf{v}_1$.

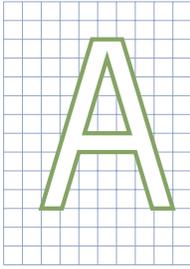


Figure 4.5.1

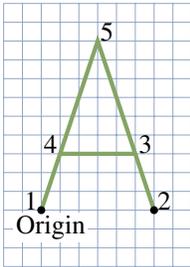


Figure 4.5.2

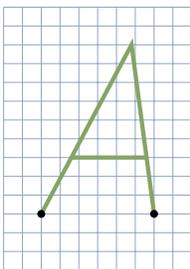


Figure 4.5.3

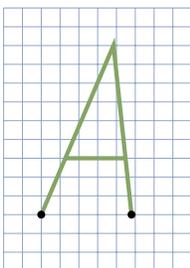


Figure 4.5.4

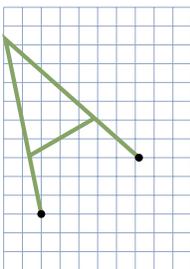


Figure 4.5.5

This simplified letter can then be stored as a data matrix

$$\begin{array}{c} \text{Vertex} \\ D = \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}$$

where the columns are the coordinates of the vertices in order. Then if we want to transform the letter by a 2×2 matrix A , we left-multiply this data matrix by A (the effect is to multiply each column by A and so transform each vertex).

For example, we can slant the letter to the right by multiplying by an x -shear matrix $A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ —see Section 2.2. The result is the letter with data matrix

$$A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 5.6 & 1.6 & 4.8 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}$$

which is shown in Figure 4.5.3.

If we want to make this slanted matrix narrower, we can now apply an x -scale matrix $B = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix}$ that shrinks the x -coordinate by 0.8. The result is the composite transformation

$$\begin{aligned} BAD &= \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4.8 & 4.48 & 1.28 & 3.84 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \end{aligned}$$

which is drawn in Figure 4.5.4.

On the other hand, we can rotate the letter about the origin through $\frac{\pi}{6}$ (or 30°) by multiplying by the matrix $R_{\frac{\pi}{2}} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$.

This gives

$$\begin{aligned} R_{\frac{\pi}{2}} &= \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 5.196 & 2.83 & -0.634 & -1.902 \\ 0 & 3 & 5.098 & 3.098 & 9.294 \end{bmatrix} \end{aligned}$$

and is plotted in Figure 4.5.5.

This poses a problem: How do we rotate at a point other than the origin? It turns out that we can do this when we have solved another more basic problem. It is clearly important to be able to translate a screen image by a fixed vector \mathbf{w} , that is apply the transformation $T_{\mathbf{w}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $T_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ for all \mathbf{v} in \mathbb{R}^2 . The problem is that these translations are not matrix transformations $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ because they do not carry $\mathbf{0}$ to $\mathbf{0}$ (unless $\mathbf{w} = \mathbf{0}$). However, there is a clever way around this.

The idea is to represent a point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a 3×1 column $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, called the **homogeneous coordinates** of \mathbf{v} . Then translation by $\mathbf{w} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be achieved by multiplying by a 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+p \\ y+q \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\mathbf{w}}(\mathbf{v}) \\ 1 \end{bmatrix}$$

Thus, by using homogeneous coordinates we can implement the translation $T_{\mathbf{w}}$ in the top two coordinates.

On the other hand, the matrix transformation induced by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also given by a 3×3 matrix:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathbf{v} \\ 1 \end{bmatrix}$$

So everything can be accomplished at the expense of using 3×3 matrices and homogeneous coordinates.

Example 4.5.1

Rotate the letter *A* in Figure 4.5.2 through $\frac{\pi}{6}$ about the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Solution. Using homogeneous coordinates for the vertices of the letter results in a data matrix with three rows:

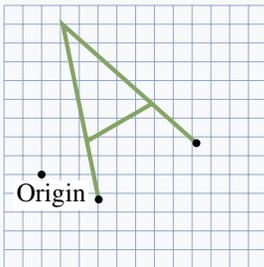


Figure 4.5.6

$$K_d = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

If we write $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, the idea is to use a composite of transformations: First translate the letter by $-\mathbf{w}$ so that the point \mathbf{w} moves to the origin, then rotate this translated letter, and then translate it by \mathbf{w} back to its original position. The matrix arithmetic is as follows (remember the order of composition!):

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3.036 & 8.232 & 5.866 & 2.402 & 1.134 \\ -1.33 & 1.67 & 3.768 & 1.768 & 7.964 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

This is plotted in Figure 4.5.6.

This discussion merely touches the surface of computer graphics, and the reader is referred to specialized books on the subject. Realistic graphic rendering requires an enormous number of matrix calculations. In fact, matrix multiplication algorithms are now embedded in microchip circuits, and can perform

over 100 million matrix multiplications per second. This is particularly important in the field of three-dimensional graphics where the homogeneous coordinates have four components and 4×4 matrices are required.

Exercises for 4.5

Exercise 4.5.1 Consider the letter A described in Figure 4.5.2. Find the data matrix for the letter obtained by:

- Rotating the letter through $\frac{\pi}{4}$ about the origin.
- Rotating the letter through $\frac{\pi}{4}$ about the point

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Exercise 4.5.2 Find the matrix for turning the letter A in Figure 4.5.2 upside-down in place.

Exercise 4.5.3 Find the 3×3 matrix for reflecting in the line $y = mx + b$. Use $\begin{bmatrix} 1 \\ m \end{bmatrix}$ as direction vector for the line.

Exercise 4.5.4 Find the 3×3 matrix for rotating through the angle θ about the point $P(a, b)$.

Exercise 4.5.5 Find the reflection of the point P in the line $y = 1 + 2x$ in \mathbb{R}^2 if:

- $P = P(1, 1)$
- $P = P(1, 4)$
- What about $P = P(1, 3)$? Explain. [*Hint*: Example 4.5.1 and Section 4.4.]

Supplementary Exercises for Chapter 4

Exercise 4.1 Suppose that \mathbf{u} and \mathbf{v} are nonzero vectors. If \mathbf{u} and \mathbf{v} are not parallel, and $a\mathbf{u} + b\mathbf{v} = a_1\mathbf{u} + b_1\mathbf{v}$, show that $a = a_1$ and $b = b_1$.

Exercise 4.2 Consider a triangle with vertices A , B , and C . Let E and F be the midpoints of sides AB and AC , respectively, and let the medians EC and FB meet at O . Write $\overrightarrow{EO} = s\overrightarrow{EC}$ and $\overrightarrow{FO} = t\overrightarrow{FB}$, where s and t are scalars. Show that $s = t = \frac{1}{3}$ by expressing \overrightarrow{AO} two ways in the form $a\overrightarrow{EO} + b\overrightarrow{AC}$, and applying Exercise 4.1. Conclude that the medians of a triangle meet at the point on each that is one-third of the way from the midpoint to the vertex (and so are concurrent).

Exercise 4.3 A river flows at 1 km/h and a swimmer moves at 2 km/h (relative to the water). At what angle must he swim to go straight across? What is his resulting speed?

Exercise 4.4 A wind is blowing from the south at 75

knots, and an airplane flies heading east at 100 knots. Find the resulting velocity of the airplane.

Exercise 4.5 An airplane pilot flies at 300 km/h in a direction 30° south of east. The wind is blowing from the south at 150 km/h.

- Find the resulting direction and speed of the airplane.
- Find the speed of the airplane if the wind is from the west (at 150 km/h).

Exercise 4.6 A rescue boat has a top speed of 13 knots. The captain wants to go due east as fast as possible in water with a current of 5 knots due south. Find the velocity vector $\mathbf{v} = (x, y)$ that she must achieve, assuming the x and y axes point east and north, respectively, and find her resulting speed.