b. Show that $U \cap W = \{\mathbf{0}\}$ if and only if $\{\mathbf{u}, \mathbf{w}\}$ is independent for any nonzero vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W.

c. If B and D are bases of U and W, and if $U \cap W = \{\mathbf{0}\}$, show that $B \cup D = \{\mathbf{v} \mid \mathbf{v}$ is in B or $D\}$ is independent.

Exercise 6.3.34 If U and W are vector spaces, let $V = \{(\mathbf{u}, \mathbf{w}) \mid \mathbf{u}$ in U and $\mathbf{w}$ in $W\}$.

a. Show that V is a vector space if $(\mathbf{u}, \mathbf{w}) + (\mathbf{u}_1, \mathbf{w}_1) = (\mathbf{u} + \mathbf{u}_1, \mathbf{w} + \mathbf{w}_1)$ and $a(\mathbf{u}, \mathbf{w}) = (a\mathbf{u}, a\mathbf{w})$.

b. If $\dim U = m$ and $\dim W = n$, show that $\dim V = m + n$.

c. If $V_1, \dots, V_m$ are vector spaces, let

$$\begin{aligned} V &= V_1 \times \cdots \times V_m \\ &= \{(\mathbf{v}_1, \dots, \mathbf{v}_m) \mid \mathbf{v}_i \in V_i \text{ for each } i\} \end{aligned}$$

denote the space of n-tuples from the V_i with componentwise operations (see Exercise 6.1.17). If $\dim V_i = n_i$ for each i, show that $\dim V = n_1 + \cdots + n_m$.

Exercise 6.3.35 Let $\mathbf{D}_n$ denote the set of all functions f from the set $\{1, 2, \dots, n\}$ to $\mathbb{R}$.

a. Show that $\mathbf{D}_n$ is a vector space with pointwise addition and scalar multiplication.

b. Show that $\{S_1, S_2, \dots, S_n\}$ is a basis of $\mathbf{D}_n$ where, for each $k = 1, 2, \dots, n$, the function S_k is defined by $S_k(k) = 1$, whereas $S_k(j) = 0$ if $j \neq k$.

Exercise 6.3.36 A polynomial $p(x)$ is called **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$. Let E_n and O_n denote the sets of even and odd polynomials in $\mathbf{P}_n$.

a. Show that E_n is a subspace of $\mathbf{P}_n$ and find $\dim E_n$.

b. Show that O_n is a subspace of $\mathbf{P}_n$ and find $\dim O_n$.

Exercise 6.3.37 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be independent in a vector space V, and let A be an $n \times n$ matrix. Define $\mathbf{u}_1, \dots, \mathbf{u}_n$ by

$$\left[\begin{array}{c} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{array}\right] = A \left[\begin{array}{c} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{array}\right]$$

(See Exercise 6.1.18.) Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is independent if and only if A is invertible.

6.4 Finite Dimensional Spaces

Up to this point, we have had no guarantee that an arbitrary vector space *has* a basis—and hence no guarantee that one can speak *at all* of the dimension of V. However, Theorem 6.4.1 will show that any space that is spanned by a finite set of vectors has a (finite) basis: The proof requires the following basic lemma, of interest in itself, that gives a way to enlarge a given independent set of vectors.

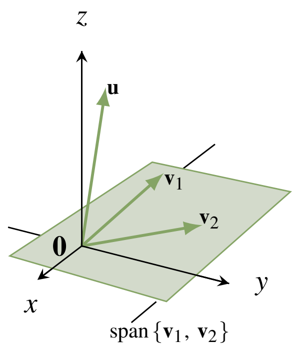
Lemma 6.4.1: Independent Lemma

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be an independent set of vectors in a vector space V. If $\mathbf{u} \in V$ but[5] *$\mathbf{u} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, then $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is also independent.*

Proof. Let $t\mathbf{u} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$; we must show that all the coefficients are zero. First, $t = 0$ because, otherwise, $\mathbf{u} = -\frac{t_1}{t}\mathbf{v}_1 - \frac{t_2}{t}\mathbf{v}_2 - \cdots - \frac{t_k}{t}\mathbf{v}_k$ is in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, contrary to our assumption.

[5] If X is a set, we write $a \in X$ to indicate that a is an element of the set X. If a is not an element of X, we write $a \notin X$.

Hence $t = 0$. But then $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ so the rest of the t_i are zero by the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. This is what we wanted. □



Note that the converse of Lemma 6.4.1 is also true: if $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent, then $\mathbf{u}$ is not in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

As an illustration, suppose that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is independent in $\mathbb{R}^3$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, so $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane through the origin (shaded in the diagram). By Lemma 6.4.1, $\mathbf{u}$ is not in this plane if and only if $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2\}$ is independent.

Definition 6.7 Finite Dimensional and Infinite Dimensional Vector Spaces

*A vector space V is called **finite dimensional** if it is spanned by a finite set of vectors. Otherwise, V is called **infinite dimensional**.*

Thus the zero vector space $\{\mathbf{0}\}$ is finite dimensional because $\{\mathbf{0}\}$ is a spanning set.

Lemma 6.4.2

Let V be a finite dimensional vector space. If U is any subspace of V, then any independent subset of U can be enlarged to a finite basis of U.

Proof. Suppose that I is an independent subset of U. If $\text{span}\, I = U$ then I is already a basis of U. If $\text{span}\, I \neq U$, choose $\mathbf{u}_1 \in U$ such that $\mathbf{u}_1 \notin \text{span}\, I$. Hence the set $I \cup \{\mathbf{u}_1\}$ is independent by Lemma 6.4.1. If $\text{span}\,(I \cup \{\mathbf{u}_1\}) = U$ we are done; otherwise choose $\mathbf{u}_2 \in U$ such that $\mathbf{u}_2 \notin \text{span}\,(I \cup \{\mathbf{u}_1\})$. Hence $I \cup \{\mathbf{u}_1, \mathbf{u}_2\}$ is independent, and the process continues. We claim that a basis of U will be reached eventually. Indeed, if no basis of U is ever reached, the process creates arbitrarily large independent sets in V. But this is impossible by the fundamental theorem because V is finite dimensional and so is spanned by a finite set of vectors. □

Theorem 6.4.1

Let V be a finite dimensional vector space spanned by m vectors.

1. *V has a finite basis, and $\dim V \leq m$.*
2. *Every independent set of vectors in V can be enlarged to a basis of V by adding vectors from any fixed basis of V.*
3. *If U is a subspace of V, then*

   a. *U is finite dimensional and $\dim U \leq \dim V$.*
   b. *If $\dim U = \dim V$ then $U = V$.*

Proof.

1. If $V = \{\mathbf{0}\}$, then V has an empty basis and $\dim V = 0 \leq m$. Otherwise, let $\mathbf{v} \neq \mathbf{0}$ be a vector in V. Then $\{\mathbf{v}\}$ is independent, so (1) follows from Lemma 6.4.2 with $U = V$.

2. We refine the proof of Lemma 6.4.2. Fix a basis B of V and let I be an independent subset of V. If $\text{span } I = V$ then I is already a basis of V. If $\text{span } I \neq V$, then B is not contained in I (because B spans V). Hence choose $\mathbf{b}_1 \in B$ such that $\mathbf{b}_1 \notin \text{span } I$. Hence the set $I \cup \{\mathbf{b}_1\}$ is independent by Lemma 6.4.1. If $\text{span}\,(I \cup \{\mathbf{b}_1\}) = V$ we are done; otherwise a similar argument shows that $(I \cup \{\mathbf{b}_1,\ \mathbf{b}_2\})$ is independent for some $\mathbf{b}_2 \in B$. Continue this process. As in the proof of Lemma 6.4.2, a basis of V will be reached eventually.

3. a. This is clear if $U = \{\mathbf{0}\}$. Otherwise, let $\mathbf{u} \neq \mathbf{0}$ in U. Then $\{\mathbf{u}\}$ can be enlarged to a finite basis B of U by Lemma 6.4.2, proving that U is finite dimensional. But B is independent in V, so $\dim U \leq \dim V$ by the fundamental theorem.

   b. This is clear if $U = \{\mathbf{0}\}$ because V *has* a basis; otherwise, it follows from (2). □

Theorem 6.4.1 shows that a vector space V is finite dimensional if and only if it has a finite basis (possibly empty), and that every subspace of a finite dimensional space is again finite dimensional.

Example 6.4.1

Enlarge the independent set $D = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ to a basis of $\mathbf{M}_{22}$.

Solution. The standard basis of $\mathbf{M}_{22}$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, so including one of these in D will produce a basis by Theorem 6.4.1. In fact including *any* of these matrices in D produces an independent set (verify), and hence a basis by Theorem 6.4.4. Of course these vectors are not the only possibilities, for example, including $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ works as well.

Example 6.4.2

Find a basis of $\mathbf{P}_3$ containing the independent set $\{1+x,\ 1+x^2\}$.

Solution. The standard basis of $\mathbf{P}_3$ is $\{1,\ x,\ x^2,\ x^3\}$, so including two of these vectors will do. If we use 1 and x^3, the result is $\{1,\ 1+x,\ 1+x^2,\ x^3\}$. This is independent because the polynomials have distinct degrees (Example 6.3.4), and so is a basis by Theorem 6.4.1. Of course, including $\{1,\ x\}$ or $\{1,\ x^2\}$ would *not* work!

Example 6.4.3

Show that the space $\mathbf{P}$ of all polynomials is infinite dimensional.

Solution. For each $n \geq 1$, $\mathbf{P}$ has a subspace $\mathbf{P}_n$ of dimension $n+1$. Suppose $\mathbf{P}$ is finite dimensional, say $\dim \mathbf{P} = m$. Then $\dim \mathbf{P}_n \leq \dim \mathbf{P}$ by Theorem 6.4.1, that is $n+1 \leq m$. This is impossible since n is arbitrary, so $\mathbf{P}$ must be infinite dimensional.

The next example illustrates how (2) of Theorem 6.4.1 can be used.

Example 6.4.4

If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k$ are independent columns in $\mathbb{R}^n$, show that they are the first k columns in some invertible $n \times n$ matrix.

Solution. By Theorem 6.4.1, expand $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k\}$ to a basis $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k, \mathbf{c}_{k+1}, \dots, \mathbf{c}_n\}$ of $\mathbb{R}^n$. Then the matrix $A = \left[\begin{array}{cccccc} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_k & \mathbf{c}_{k+1} & \dots & \mathbf{c}_n \end{array} \right]$ with this basis as its columns is an $n \times n$ matrix and it is invertible by Theorem 5.2.3.

Theorem 6.4.2

Let U and W be subspaces of the finite dimensional space V.

1. *If $U \subseteq W$, then $\dim U \leq \dim W$.*
2. *If $U \subseteq W$ and $\dim U = \dim W$, then $U = W$.*

Proof. Since W is finite dimensional, (1) follows by taking $V = W$ in part (3) of Theorem 6.4.1. Now assume $\dim U = \dim W = n$, and let B be a basis of U. Then B is an independent set in W. If $U \neq W$, then $\operatorname{span} B \neq W$, so B can be extended to an independent set of $n+1$ vectors in W by Lemma 6.4.1. This contradicts the fundamental theorem (Theorem 6.3.2) because W is spanned by $\dim W = n$ vectors. Hence $U = W$, proving (2). □

Theorem 6.4.2 is very useful. This was illustrated in Example 5.2.13 for $\mathbb{R}^2$ and $\mathbb{R}^3$; here is another example.

Example 6.4.5

If a is a number, let W denote the subspace of all polynomials in $\mathbf{P}_n$ that have a as a root:

$$W = \{p(x) \mid p(x) \in \mathbf{P}_n \text{ and } p(a) = 0\}$$

Show that $\{(x-a), (x-a)^2, \dots, (x-a)^n\}$ is a basis of W.

Solution. Observe first that $(x-a), (x-a)^2, \dots, (x-a)^n$ are members of W, and that they are independent because they have distinct degrees (Example 6.3.4). Write

$$U = \operatorname{span}\{(x-a), (x-a)^2, \dots, (x-a)^n\}$$

Then we have $U \subseteq W \subseteq \mathbf{P}_n$, $\dim U = n$, and $\dim \mathbf{P}_n = n+1$. Hence $n \leq \dim W \leq n+1$ by Theorem 6.4.2. Since $\dim W$ is an integer, we must have $\dim W = n$ or $\dim W = n+1$. But then $W = U$ or $W = \mathbf{P}_n$, again by Theorem 6.4.2. Because $W \neq \mathbf{P}_n$, it follows that $W = U$, as required.

A set of vectors is called **dependent** if it is *not* independent, that is if some nontrivial linear combination vanishes. The next result is a convenient test for dependence.

Lemma 6.4.3: Dependent Lemma

A set $D = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \dots,\ \mathbf{v}_k\}$ of vectors in a vector space V is dependent if and only if some vector in D is a linear combination of the others.

Proof. Let $\mathbf{v}_2$ (say) be a linear combination of the rest: $\mathbf{v}_2 = s_1\mathbf{v}_1 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k$. Then

$$s_1\mathbf{v}_1 + (-1)\mathbf{v}_2 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k = \mathbf{0}$$

is a nontrivial linear combination that vanishes, so D is dependent. Conversely, if D is dependent, let $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$ where some coefficient is nonzero. If (say) $t_2 \neq 0$, then $\mathbf{v}_2 = -\frac{t_1}{t_2}\mathbf{v}_1 - \frac{t_3}{t_2}\mathbf{v}_3 - \cdots - \frac{t_k}{t_2}\mathbf{v}_k$ is a linear combination of the others. □

Lemma 6.4.1 gives a way to enlarge independent sets to a basis; by contrast, Lemma 6.4.3 shows that spanning sets can be cut down to a basis.

Theorem 6.4.3

Let V be a finite dimensional vector space. Any spanning set for V can be cut down (by deleting vectors) to a basis of V.

Proof. Since V is finite dimensional, it has a finite spanning set S. Among all spanning sets contained in S, choose S_0 containing the smallest number of vectors. It suffices to show that S_0 is independent (then S_0 is a basis, proving the theorem). Suppose, on the contrary, that S_0 is not independent. Then, by Lemma 6.4.3, some vector $\mathbf{u} \in S_0$ is a linear combination of the set $S_1 = S_0 \setminus \{\mathbf{u}\}$ of vectors in S_0 other than $\mathbf{u}$. It follows that $\text{span}\, S_0 = \text{span}\, S_1$, that is, $V = \text{span}\, S_1$. But S_1 has fewer elements than S_0 so this contradicts the choice of S_0. Hence S_0 is independent after all. □

Note that, with Theorem 6.4.1, Theorem 6.4.3 completes the promised proof of Theorem 5.2.6 for the case $V = \mathbb{R}^n$.

Example 6.4.6

Find a basis of $\mathbf{P}_3$ in the spanning set $S = \{1,\ x + x^2,\ 2x - 3x^2,\ 1 + 3x - 2x^2,\ x^3\}$.

Solution. Since $\dim \mathbf{P}_3 = 4$, we must eliminate one polynomial from S. It cannot be x^3 because the span of the rest of S is contained in $\mathbf{P}_2$. But eliminating $1 + 3x - 2x^2$ does leave a basis (verify). Note that $1 + 3x - 2x^2$ is the sum of the first three polynomials in S.

Theorems 6.4.1 and 6.4.3 have other useful consequences.

Theorem 6.4.4

Let V be a vector space with $\dim V = n$, and suppose S is a set of exactly n vectors in V. Then S is independent if and only if S spans V.

Proof. Assume first that S is independent. By Theorem 6.4.1, S is contained in a basis B of V. Hence $|S| = n = |B|$ so, since $S \subseteq B$, it follows that $S = B$. In particular S spans V.

Conversely, assume that S spans V, so S contains a basis B by Theorem 6.4.3. Again $|S| = n = |B|$ so, since $S \supseteq B$, it follows that $S = B$. Hence S is independent. □

One of independence or spanning is often easier to establish than the other when showing that a set of vectors is a basis. For example if $V = \mathbb{R}^n$ it is easy to check whether a subset S of $\mathbb{R}^n$ is orthogonal (hence independent) but checking spanning can be tedious. Here are three more examples.

Example 6.4.7

Consider the set $S = \{p_0(x),\ p_1(x),\ \dots,\ p_n(x)\}$ of polynomials in $\mathbf{P}_n$. If $\deg p_k(x) = k$ for each k, show that S is a basis of $\mathbf{P}_n$.

Solution. The set S is independent—the degrees are distinct—see Example 6.3.4. Hence S is a basis of $\mathbf{P}_n$ by Theorem 6.4.4 because $\dim \mathbf{P}_n = n+1$.

Example 6.4.8

Let V denote the space of all symmetric 2×2 matrices. Find a basis of V consisting of invertible matrices.

Solution. We know that $\dim V = 3$ (Example 6.3.11), so what is needed is a set of three invertible, symmetric matrices that (using Theorem 6.4.4) is either independent or spans V. The set $\left\{ \left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{rr} 1 & 0 \\ 0 & -1 \end{array} \right], \left[\begin{array}{rr} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}$ is independent (verify) and so is a basis of the required type.

Example 6.4.9

Let A be any $n \times n$ matrix. Show that there exist n^2+1 scalars $a_0,\ a_1,\ a_2,\ \dots,\ a_{n^2}$ not all zero, such that

$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n^2} A^{n^2} = 0$$

where I denotes the $n \times n$ identity matrix.

Solution. The space $\mathbf{M}_{nn}$ of all $n \times n$ matrices has dimension n^2 by Example 6.3.7. Hence the n^2+1 matrices $I,\ A,\ A^2,\ \dots,\ A^{n^2}$ cannot be independent by Theorem 6.4.4, so a nontrivial linear combination vanishes. This is the desired conclusion.

The result in Example 6.4.9 can be written as $f(A) = 0$ where $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n^2} x^{n^2}$. In other words, A satisfies a nonzero polynomial $f(x)$ of degree at most n^2. In fact we know that A satisfies a nonzero polynomial of degree n (this is the Cayley-Hamilton theorem—see Theorem 8.7.10), but the brevity of the solution in Example 6.4.6 is an indication of the power of these methods.

If U and W are subspaces of a vector space V, there are two related subspaces that are of interest, their **sum** $U+W$ and their **intersection** $U \cap W$, defined by

$$U+W = \{\mathbf{u}+\mathbf{w} \mid \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$$
$$U \cap W = \{\mathbf{v} \in V \mid \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$$

It is routine to verify that these are indeed subspaces of V, that $U \cap W$ is contained in both U and W, and that $U+W$ contains both U and W. We conclude this section with a useful fact about the dimensions of these spaces. The proof is a good illustration of how the theorems in this section are used.

Theorem 6.4.5

Suppose that U and W are finite dimensional subspaces of a vector space V. Then $U+W$ is finite dimensional and

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Proof. Since $U \cap W \subseteq U$, it has a finite basis, say $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$. Extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{u}_1, \dots, \mathbf{u}_m\}$ of U by Theorem 6.4.1. Similarly extend $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$ to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{w}_1, \dots, \mathbf{w}_p\}$ of W. Then

$$U+W = \text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_p\}$$

as the reader can verify, so $U+W$ is finite dimensional. For the rest, it suffices to show that $\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_p\}$ is independent (verify). Suppose that

$$r_1\mathbf{x}_1 + \cdots + r_d\mathbf{x}_d + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m + t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p = \mathbf{0} \tag{6.1}$$

where the r_i, s_j, and t_k are scalars. Then

$$r_1\mathbf{x}_1 + \cdots + r_d\mathbf{x}_d + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m = -(t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p)$$

is in U (left side) and also in W (right side), and so is in $U \cap W$. Hence $(t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p)$ is a linear combination of $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$, so $t_1 = \cdots = t_p = 0$, because $\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{w}_1, \dots, \mathbf{w}_p\}$ is independent. Similarly, $s_1 = \cdots = s_m = 0$, so (6.1) becomes $r_1\mathbf{x}_1 + \cdots + r_d\mathbf{x}_d = \mathbf{0}$. It follows that $r_1 = \cdots = r_d = 0$, as required. □

Theorem 6.4.5 is particularly interesting if $U \cap W = \{\mathbf{0}\}$. Then there are *no* vectors $\mathbf{x}_i$ in the above proof, and the argument shows that if $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ are bases of U and W respectively, then $\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_p\}$ is a basis of $U+W$. In this case $U+W$ is said to be a **direct sum** (written $U \oplus W$); we return to this in Chapter 9.

Exercises for 6.4

Exercise 6.4.1 In each case, find a basis for V that includes the vector $\mathbf{v}$.

a. $V = \mathbb{R}^3$, $\mathbf{v} = (1, -1, 1)$

b. $V = \mathbb{R}^3$, $\mathbf{v} = (0, 1, 1)$

c. $V = \mathbf{M}_{22}$, $\mathbf{v} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$

d. $V = \mathbf{P}_2$, $\mathbf{v} = x^2 - x + 1$

Exercise 6.4.2 In each case, find a basis for V among the given vectors.

a. $V = \mathbb{R}^3$,
$\{(1, 1, -1), (2, 0, 1), (-1, 1, -2), (1, 2, 1)\}$

b. $V = \mathbf{P}_2$, $\{x^2 + 3, x + 2, x^2 - 2x - 1, x^2 + x\}$

Exercise 6.4.3 In each case, find a basis of V containing $\mathbf{v}$ and $\mathbf{w}$.

a. $V = \mathbb{R}^4$, $\mathbf{v} = (1, -1, 1, -1)$, $\mathbf{w} = (0, 1, 0, 1)$

b. $V = \mathbb{R}^4$, $\mathbf{v} = (0, 0, 1, 1)$, $\mathbf{w} = (1, 1, 1, 1)$

c. $V = \mathbf{M}_{22}$, $\mathbf{v} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, $\mathbf{w} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$

d. $V = \mathbf{P}_3$, $\mathbf{v} = x^2 + 1$, $\mathbf{w} = x^2 + x$

Exercise 6.4.4

a. If z is not a real number, show that $\{z, z^2\}$ is a basis of the real vector space $\mathbb{C}$ of all complex numbers.

b. If z is neither real nor pure imaginary, show that $\{z, \overline{z}\}$ is a basis of $\mathbb{C}$.

Exercise 6.4.5 In each case use Theorem 6.4.4 to decide if S is a basis of V.

a. $V = \mathbf{M}_{22}$;
$S = \left\{ \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}$

b. $V = \mathbf{P}_3$; $S = \{2x^2, 1 + x, 3, 1 + x + x^2 + x^3\}$

Exercise 6.4.6

a. Find a basis of $\mathbf{M}_{22}$ consisting of matrices with the property that $A^2 = A$.

b. Find a basis of $\mathbf{P}_3$ consisting of polynomials whose coefficients sum to 4. What if they sum to 0?

Exercise 6.4.7 If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis of V, determine which of the following are bases.

a. $\{\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{w}, \mathbf{v} + \mathbf{w}\}$

b. $\{2\mathbf{u} + \mathbf{v} + 3\mathbf{w}, 3\mathbf{u} + \mathbf{v} - \mathbf{w}, \mathbf{u} - 4\mathbf{w}\}$

c. $\{\mathbf{u}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$

d. $\{\mathbf{u}, \mathbf{u} + \mathbf{w}, \mathbf{u} - \mathbf{w}, \mathbf{v} + \mathbf{w}\}$

Exercise 6.4.8

a. Can two vectors span $\mathbb{R}^3$? Can they be linearly independent? Explain.

b. Can four vectors span $\mathbb{R}^3$? Can they be linearly independent? Explain.

Exercise 6.4.9 Show that any nonzero vector in a finite dimensional vector space is part of a basis.

Exercise 6.4.10 If A is a square matrix, show that $\det A = 0$ if and only if some row is a linear combination of the others.

Exercise 6.4.11 Let D, I, and X denote finite, nonempty sets of vectors in a vector space V. Assume that D is dependent and I is independent. In each case answer yes or no, and defend your answer.

a. If $X \supseteq D$, must X be dependent?

b. If $X \subseteq D$, must X be dependent?

c. If $X \supseteq I$, must X be independent?

d. If $X \subseteq I$, must X be independent?

Exercise 6.4.12 If U and W are subspaces of V and $\dim U = 2$, show that either $U \subseteq W$ or $\dim(U \cap W) \leq 1$.

Exercise 6.4.13 Let A be a nonzero 2×2 matrix and write $U = \{X \text{ in } \mathbf{M}_{22} \mid XA = AX\}$. Show that $\dim U \geq 2$. [*Hint*: I and A are in U.]

Exercise 6.4.14 If $U \subseteq \mathbb{R}^2$ is a subspace, show that $U = \{\mathbf{0}\}$, $U = \mathbb{R}^2$, or U is a line through the origin.

Exercise 6.4.15 Given $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_k$, and $\mathbf{v}$, let $U = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ and $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}\}$. Show that either $\dim W = \dim U$ or $\dim W = 1 + \dim U$.

Exercise 6.4.16 Suppose U is a subspace of $\mathbf{P}_1$, $U \neq \{0\}$, and $U \neq \mathbf{P}_1$. Show that either $U = \mathbb{R}$ or $U = \mathbb{R}(a + x)$ for some a in $\mathbb{R}$.

Exercise 6.4.17 Let U be a subspace of V and assume $\dim V = 4$ and $\dim U = 2$. Does every basis of V result from adding (two) vectors to some basis of U? Defend your answer.

Exercise 6.4.18 Let U and W be subspaces of a vector space V.

a. If $\dim V = 3$, $\dim U = \dim W = 2$, and $U \neq W$, show that $\dim(U \cap W) = 1$.

b. Interpret (a.) geometrically if $V = \mathbb{R}^3$.

Exercise 6.4.19 Let $U \subseteq W$ be subspaces of V with $\dim U = k$ and $\dim W = m$, where $k < m$. If $k < l < m$, show that a subspace X exists where $U \subseteq X \subseteq W$ and $\dim X = l$.

Exercise 6.4.20 Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a *maximal* independent set in a vector space V. That is, no set of more than n vectors S is independent. Show that B is a basis of V.

Exercise 6.4.21 Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a *minimal* spanning set for a vector space V. That is, V cannot be spanned by fewer than n vectors. Show that B is a basis of V.

Exercise 6.4.22

a. Let $p(x)$ and $q(x)$ lie in $\mathbf{P}_1$ and suppose that $p(1) \neq 0$, $q(2) \neq 0$, and $p(2) = 0 = q(1)$. Show that $\{p(x), q(x)\}$ is a basis of $\mathbf{P}_1$. [*Hint*: If $rp(x) + sq(x) = 0$, evaluate at $x = 1$, $x = 2$.]

b. Let $B = \{p_0(x), p_1(x), \dots, p_n(x)\}$ be a set of polynomials in $\mathbf{P}_n$. Assume that there exist numbers $a_0, a_1, \dots, a_n$ such that $p_i(a_i) \neq 0$ for each i but $p_i(a_j) = 0$ if i is different from j. Show that B is a basis of $\mathbf{P}_n$.

Exercise 6.4.23 Let V be the set of all infinite sequences $(a_0, a_1, a_2, \dots)$ of real numbers. Define addition and scalar multiplication by

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (a_0 + b_0, a_1 + b_1, \dots)$$

and

$$r(a_0, a_1, \dots) = (ra_0, ra_1, \dots)$$

a. Show that V is a vector space.

b. Show that V is not finite dimensional.

c. [For those with some calculus.] Show that the set of convergent sequences (that is, $\lim\limits_{n \to \infty} a_n$ exists) is a subspace, also of infinite dimension.

Exercise 6.4.24 Let A be an $n \times n$ matrix of rank r. If $U = \{X \text{ in } \mathbf{M}_{nn} \mid AX = 0\}$, show that $\dim U = n(n - r)$. [*Hint*: Exercise 6.3.34.]

Exercise 6.4.25 Let U and W be subspaces of V.

a. Show that $U + W$ is a subspace of V containing both U and W.

b. Show that $\text{span}\{\mathbf{u}, \mathbf{w}\} = \mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{w}$ for any vectors $\mathbf{u}$ and $\mathbf{w}$.

c. Show that

$$\begin{aligned} &\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_n\} \\ &= \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_m\} + \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_n\} \end{aligned}$$

for any vectors $\mathbf{u}_i$ in U and $\mathbf{w}_j$ in W.

Exercise 6.4.26 If A and B are $m \times n$ matrices, show that $\text{rank}(A + B) \leq \text{rank} A + \text{rank} B$. [*Hint*: If U and V are the column spaces of A and B, respectively, show that the column space of $A + B$ is contained in $U + V$ and that $\dim(U + V) \leq \dim U + \dim V$. (See Theorem 6.4.5.)]

6.5 An Application to Polynomials

The vector space of all polynomials of degree at most n is denoted $\mathbf{P}_n$, and it was established in Section 6.3 that $\mathbf{P}_n$ has dimension $n+1$; in fact, $\{1,\ x,\ x^2,\ \dots,\ x^n\}$ is a basis. More generally, *any* $n+1$ polynomials of distinct degrees form a basis, by Theorem 6.4.4 (they are independent by Example 6.3.4). This proves

Theorem 6.5.1

Let $p_0(x),\ p_1(x),\ p_2(x),\ \dots,\ p_n(x)$ be polynomials in $\mathbf{P}_n$ of degrees $0,\ 1,\ 2,\ \dots,\ n$, respectively. Then $\{p_0(x),\ \dots,\ p_n(x)\}$ is a basis of $\mathbf{P}_n$.

An immediate consequence is that $\{1,\ (x-a),\ (x-a)^2,\ \dots,\ (x-a)^n\}$ is a basis of $\mathbf{P}_n$ for any number a. Hence we have the following:

Corollary 6.5.1

If a is any number, every polynomial $f(x)$ of degree at most n has an expansion in powers of $(x-a)$:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n \tag{6.2}$$

If $f(x)$ is evaluated at $x = a$, then equation (6.2) becomes

$$f(x) = a_0 + a_1(a-a) + \cdots + a_n(a-a)^n = a_0$$

Hence $a_0 = f(a)$, and equation (6.2) can be written $f(x) = f(a) + (x-a)g(x)$, where $g(x)$ is a polynomial of degree $n-1$ (this assumes that $n \geq 1$). If it happens that $f(a) = 0$, then it is clear that $f(x)$ has the form $f(x) = (x-a)g(x)$. Conversely, every such polynomial certainly satisfies $f(a) = 0$, and we obtain:

Corollary 6.5.2

Let $f(x)$ be a polynomial of degree $n \geq 1$ and let a be any number. Then:

Remainder Theorem

1. *$f(x) = f(a) + (x-a)g(x)$ for some polynomial $g(x)$ of degree $n-1$.*

Factor Theorem

2. *$f(a) = 0$ if and only if $f(x) = (x-a)g(x)$ for some polynomial $g(x)$.*

The polynomial $g(x)$ can be computed easily by using "long division" to divide $f(x)$ by $(x-a)$—see Appendix D.

All the coefficients in the expansion (6.2) of $f(x)$ in powers of $(x-a)$ can be determined in terms of the derivatives of $f(x)$.[6] These will be familiar to students of calculus. Let $f^{(n)}(x)$ denote the nth derivative

[6]The discussion of Taylor's theorem can be omitted with no loss of continuity.