Exercise 7.1.22 If $T : \mathbf{M}_{nn} \to \mathbb{R}$ is any linear transformation satisfying $T(AB) = T(BA)$ for all A and B in $\mathbf{M}_{nn}$, show that there exists a number k such that $T(A) = k \operatorname{tr} A$ for all A. (See Lemma 5.5.1.) [*Hint*: Let E_{ij} denote the $n \times n$ matrix with 1 in the (i, j) position and zeros elsewhere.

Show that $E_{ik}E_{lj} = \begin{cases} 0 & \text{if } k \neq l \\ E_{ij} & \text{if } k = l \end{cases}$. Use this to show that $T(E_{ij}) = 0$ if $i \neq j$ and $T(E_{11}) = T(E_{22}) = \cdots = T(E_{nn})$. Put $k = T(E_{11})$ and use the fact that $\{E_{ij} \mid 1 \leq i, j \leq n\}$ is a basis of $\mathbf{M}_{nn}$.]

Exercise 7.1.23 Let $T : \mathbb{C} \to \mathbb{C}$ be a linear transformation of the real vector space $\mathbb{C}$ and assume that $T(a) = a$ for every real number a. Show that the following are equivalent:

a. $T(zw) = T(z)T(w)$ for all z and w in $\mathbb{C}$.

b. Either $T = 1_{\mathbb{C}}$ or $T(z) = \overline{z}$ for each z in $\mathbb{C}$ (where $\overline{z}$ denotes the conjugate).

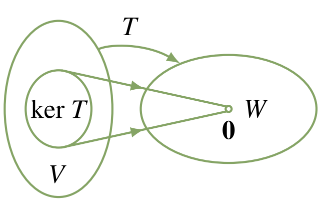
7.2 Kernel and Image of a Linear Transformation

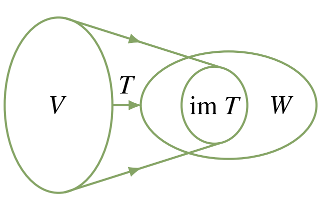
This section is devoted to two important subspaces associated with a linear transformation $T : V \to W$.

Definition 7.2 Kernel and Image of a Linear Transformation

*The **kernel** of T (denoted $\ker T$) and the **image** of T (denoted $\operatorname{im} T$ or $T(V)$) are defined by*

$$\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$$
$$\operatorname{im} T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$$





The kernel of T is often called the **nullspace** of T because it consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$. The image of T is often called the **range** of T and consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. These subspaces are depicted in the diagrams.

Example 7.2.1

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by the $m \times n$ matrix A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. Then

$$\ker T_A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \operatorname{null} A \quad \text{and}$$
$$\operatorname{im} T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \operatorname{im} A$$

Hence the following theorem extends Example 5.1.2.

Theorem 7.2.1

Let $T : V \to W$ be a linear transformation.

1. $\ker T$ *is a subspace of* V.
2. $\operatorname{im} T$ *is a subspace of* W.

Proof. The fact that $T(\mathbf{0}) = \mathbf{0}$ shows that $\ker T$ and $\operatorname{im} T$ contain the zero vector of V and W respectively.

1. If $\mathbf{v}$ and $\mathbf{v}_1$ lie in $\ker T$, then $T(\mathbf{v}) = \mathbf{0} = T(\mathbf{v}_1)$, so

$$T(\mathbf{v} + \mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$
$$T(r\mathbf{v}) = rT(\mathbf{v}) = r\mathbf{0} = \mathbf{0} \quad \text{for all } r \text{ in } \mathbb{R}$$

Hence $\mathbf{v} + \mathbf{v}_1$ and $r\mathbf{v}$ lie in $\ker T$ (they satisfy the required condition), so $\ker T$ is a subspace of V by the subspace test (Theorem 6.2.1).

2. If $\mathbf{w}$ and $\mathbf{w}_1$ lie in $\operatorname{im} T$, write $\mathbf{w} = T(\mathbf{v})$ and $\mathbf{w}_1 = T(\mathbf{v}_1)$ where $\mathbf{v}, \mathbf{v}_1 \in V$. Then

$$\mathbf{w} + \mathbf{w}_1 = T(\mathbf{v}) + T(\mathbf{v}_1) = T(\mathbf{v} + \mathbf{v}_1)$$
$$r\mathbf{w} = rT(\mathbf{v}) = T(r\mathbf{v}) \quad \text{for all } r \text{ in } \mathbb{R}$$

Hence $\mathbf{w} + \mathbf{w}_1$ and $r\mathbf{w}$ both lie in $\operatorname{im} T$ (they have the required form), so $\operatorname{im} T$ is a subspace of W.

□

Given a linear transformation $T : V \to W$:

$\dim(\ker T)$ is called the **nullity** of T and denoted as $\operatorname{nullity}(T)$
$\dim(\operatorname{im} T)$ is called the **rank** of T and denoted as $\operatorname{rank}(T)$

The rank of a matrix A was defined earlier to be the dimension of $\operatorname{col} A$, the column space of A. The two usages of the word *rank* are consistent in the following sense. Recall the definition of T_A in Example 7.2.1.

Example 7.2.2

Given an $m \times n$ matrix A, show that $\operatorname{im} T_A = \operatorname{col} A$, so $\operatorname{rank} T_A = \operatorname{rank} A$.

Solution. Write $A = \left[\begin{array}{ccc} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{array} \right]$ in terms of its columns. Then

$$\operatorname{im} T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \{x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n \mid x_i \text{ in } \mathbb{R}\}$$

using Definition 2.5. Hence $\operatorname{im} T_A$ is the column space of A; the rest follows.

Often, a useful way to study a subspace of a vector space is to exhibit it as the kernel or image of a linear transformation. Here is an example.

Example 7.2.3

Define a transformation $P : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $P(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Show that P is linear and that:

a. $\ker P$ consists of all symmetric matrices.

b. $\operatorname{im} P$ consists of all skew-symmetric matrices.

Solution. The verification that P is linear is left to the reader. To prove part (a), note that a matrix A lies in $\ker P$ just when $0 = P(A) = A - A^T$, and this occurs if and only if $A = A^T$—that is, A is symmetric. Turning to part (b), the space $\operatorname{im} P$ consists of all matrices $P(A)$, A in $\mathbf{M}_{nn}$. Every such matrix is skew-symmetric because

$$P(A)^T = (A - A^T)^T = A^T - A = -P(A)$$

On the other hand, if S is skew-symmetric (that is, $S^T = -S$), then S lies in $\operatorname{im} P$. In fact,

$$P\left[\tfrac{1}{2}S\right] = \tfrac{1}{2}S - \left[\tfrac{1}{2}S\right]^T = \tfrac{1}{2}(S - S^T) = \tfrac{1}{2}(S + S) = S$$

One-to-One and Onto Transformations

Definition 7.3 One-to-one and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation.

1. *T is said to be* **onto** *if $\operatorname{im} T = W$.*
2. *T is said to be* **one-to-one** *if $T(\mathbf{v}) = T(\mathbf{v}_1)$ implies $\mathbf{v} = \mathbf{v}_1$.*

A vector $\mathbf{w}$ in W is said to be **hit** by T if $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V. Then T is onto if every vector in W is hit at least once, and T is one-to-one if no element of W gets hit twice. Clearly the onto transformations T are those for which $\operatorname{im} T = W$ is as large a subspace of W as possible. By contrast, Theorem 7.2.2 shows that the one-to-one transformations T are the ones with $\ker T$ as *small* a subspace of V as possible.

Theorem 7.2.2

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Proof. If T is one-to-one, let $\mathbf{v}$ be any vector in $\ker T$. Then $T(\mathbf{v}) = \mathbf{0}$, so $T(\mathbf{v}) = T(\mathbf{0})$. Hence $\mathbf{v} = \mathbf{0}$ because T is one-to-one. Hence $\ker T = \{\mathbf{0}\}$.

Conversely, assume that $\ker T = \{\mathbf{0}\}$ and let $T(\mathbf{v}) = T(\mathbf{v}_1)$ with $\mathbf{v}$ and $\mathbf{v}_1$ in V. Then $T(\mathbf{v} - \mathbf{v}_1) = T(\mathbf{v}) - T(\mathbf{v}_1) = \mathbf{0}$, so $\mathbf{v} - \mathbf{v}_1$ lies in $\ker T = \{\mathbf{0}\}$. This means that $\mathbf{v} - \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v} = \mathbf{v}_1$, proving that T is one-to-one. □

Example 7.2.4

The identity transformation $1_V : V \to V$ is both one-to-one and onto for any vector space V.

Example 7.2.5

Consider the linear transformations

$$\begin{aligned} S : \mathbb{R}^3 \to \mathbb{R}^2 &\quad \text{given by } S(x,\ y,\ z) = (x+y,\ x-y) \\ T : \mathbb{R}^2 \to \mathbb{R}^3 &\quad \text{given by } T(x,\ y) = (x+y,\ x-y,\ x) \end{aligned}$$

Show that T is one-to-one but not onto, whereas S is onto but not one-to-one.

Solution. The verification that they are linear is omitted. T is one-to-one because

$$\ker T = \{(x,\ y) \mid x+y = x-y = x = 0\} = \{(0,\ 0)\}$$

However, it is not onto. For example $(0,\ 0,\ 1)$ does not lie in $\operatorname{im} T$ because if $(0,\ 0,\ 1) = (x+y,\ x-y,\ x)$ for some x and y, then $x+y = 0 = x-y$ and $x = 1$, an impossibility. Turning to S, it is not one-to-one by Theorem 7.2.2 because $(0,\ 0,\ 1)$ lies in $\ker S$. But every element $(s,\ t)$ in $\mathbb{R}^2$ lies in $\operatorname{im} S$ because $(s,\ t) = (x+y,\ x-y) = S(x,\ y,\ z)$ for some x, y, and z (in fact, $x = \frac{1}{2}(s+t)$, $y = \frac{1}{2}(s-t)$, and $z = 0$). Hence S is onto.

Example 7.2.6

Let U be an invertible $m \times m$ matrix and define

$$T : \mathbf{M}_{mn} \to \mathbf{M}_{mn} \quad \text{by} \quad T(X) = UX \text{ for all } X \text{ in } \mathbf{M}_{mn}$$

Show that T is a linear transformation that is both one-to-one and onto.

Solution. The verification that T is linear is left to the reader. To see that T is one-to-one, let $T(X) = 0$. Then $UX = 0$, so left-multiplication by U^{-1} gives $X = 0$. Hence $\ker T = \{\mathbf{0}\}$, so T is one-to-one. Finally, if Y is any member of $\mathbf{M}_{mn}$, then $U^{-1}Y$ lies in $\mathbf{M}_{mn}$ too, and $T(U^{-1}Y) = U(U^{-1}Y) = Y$. This shows that T is onto.

The linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ all have the form T_A for some $m \times n$ matrix A (Theorem 2.6.2). The next theorem gives conditions under which they are onto or one-to-one. Note the connection with Theorem 5.4.3 and Theorem 5.4.4.

Theorem 7.2.3

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

1. *T_A is onto if and only if* $\operatorname{rank} A = m$.
2. *T_A is one-to-one if and only if* $\operatorname{rank} A = n$.

Proof.

1. We have that $\operatorname{im} T_A$ is the column space of A (see Example 7.2.2), so T_A is onto if and only if the column space of A is $\mathbb{R}^m$. Because the rank of A is the dimension of the column space, this holds if and only if $\operatorname{rank} A = m$.
2. $\ker T_A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, so (using Theorem 7.2.2) T_A is one-to-one if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. This is equivalent to $\operatorname{rank} A = n$ by Theorem 5.4.3. □

The Dimension Theorem

Let A denote an $m \times n$ matrix of rank r and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ denote the corresponding matrix transformation given by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. It follows from Example 7.2.1 and Example 7.2.2 that $\operatorname{im} T_A = \operatorname{col} A$, so $\dim(\operatorname{im} T_A) = \dim(\operatorname{col} A) = r$. On the other hand Theorem 5.4.2 shows that $\dim(\ker T_A) = \dim(\operatorname{null} A) = n - r$. Combining these we see that

$$\dim(\operatorname{im} T_A) + \dim(\ker T_A) = n \quad \text{for every } m \times n \text{ matrix } A$$

The main result of this section is a deep generalization of this observation.

Theorem 7.2.4: Dimension Theorem

Let $T : V \to W$ be any linear transformation and assume that $\ker T$ and $\operatorname{im} T$ are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

In other words, $\dim V = \operatorname{nullity}(T) + \operatorname{rank}(T)$.

Proof. Every vector in $\operatorname{im} T = T(V)$ has the form $T(\mathbf{v})$ for some $\mathbf{v}$ in V. Hence let $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_r)\}$ be a basis of $\operatorname{im} T$, where the $\mathbf{e}_i$ lie in V. Let $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\}$ be any basis of $\ker T$. Then $\dim(\operatorname{im} T) = r$ and $\dim(\ker T) = k$, so it suffices to show that $B = \{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{f}_1, \dots, \mathbf{f}_k\}$ is a basis of V.

1. *B spans V.* If $\mathbf{v}$ lies in V, then $T(\mathbf{v})$ lies in $\operatorname{im} V$, so

$$T(\mathbf{v}) = t_1 T(\mathbf{e}_1) + t_2 T(\mathbf{e}_2) + \cdots + t_r T(\mathbf{e}_r) \quad t_i \text{ in } \mathbb{R}$$

This implies that $\mathbf{v} - t_1\mathbf{e}_1 - t_2\mathbf{e}_2 - \cdots - t_r\mathbf{e}_r$ lies in $\ker T$ and so is a linear combination of $\mathbf{f}_1, \dots, \mathbf{f}_k$. Hence $\mathbf{v}$ is a linear combination of the vectors in B.

2. B is *linearly independent*. Suppose that t_i and s_j in $\mathbb{R}$ satisfy

$$t_1\mathbf{e}_1 + \cdots + t_r\mathbf{e}_r + s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0} \tag{7.1}$$

Applying T gives $t_1T(\mathbf{e}_1) + \cdots + t_rT(\mathbf{e}_r) = \mathbf{0}$ (because $T(\mathbf{f}_i) = \mathbf{0}$ for each i). Hence the independence of $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ yields $t_1 = \cdots = t_r = 0$. But then (7.1) becomes

$$s_1\mathbf{f}_1 + \cdots + s_k\mathbf{f}_k = \mathbf{0}$$

so $s_1 = \cdots = s_k = 0$ by the independence of $\{\mathbf{f}_1, \dots, \mathbf{f}_k\}$. This proves that B is linearly independent.

□

Note that the vector space V is not assumed to be finite dimensional in Theorem 7.2.4. In fact, verifying that $\ker T$ and $\operatorname{im} T$ are both finite dimensional is often an important way to *prove* that V is finite dimensional.

Note further that $r + k = n$ in the proof so, after relabelling, we end up with a basis

$$B = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \dots,\ \mathbf{e}_r,\ \mathbf{e}_{r+1},\ \dots,\ \mathbf{e}_n\}$$

of V with the property that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$ and $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$. In fact, if V is known in advance to be finite dimensional, then *any* basis $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of $\ker T$ can be extended to a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ of V by Theorem 6.4.1. Moreover, it turns out that, no matter how this is done, the vectors $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ will be a basis of $\operatorname{im} T$. This result is useful, and we record it for reference. The proof is much like that of Theorem 7.2.4 and is left as Exercise 7.2.26.

Theorem 7.2.5

Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_n\}$ is a basis of $\ker T$. Then $\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_r)\}$ is a basis of $\operatorname{im} T$, and hence $r = \operatorname{rank} T$.

The dimension theorem is one of the most useful results in all of linear algebra. It shows that if either $\dim(\ker T)$ or $\dim(\operatorname{im} T)$ can be found, then the other is automatically known. In many cases it is easier to compute one than the other, so the theorem is a real asset. The rest of this section is devoted to illustrations of this fact. The next example uses the dimension theorem to give a different proof of the first part of Theorem 5.4.2.

Example 7.2.7

Let A be an $m \times n$ matrix of rank r. Show that the space $\operatorname{null} A$ of all solutions of the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables has dimension $n - r$.

Solution. The space in question is just $\ker T_A$, where $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. But $\dim(\operatorname{im} T_A) = \operatorname{rank} T_A = \operatorname{rank} A = r$ by Example 7.2.2, so $\dim(\ker T_A) = n - r$ by the dimension theorem.

Example 7.2.8

If $T : V \to W$ is a linear transformation where V is finite dimensional, then

$$\dim(\ker T) \leq \dim V \quad \text{and} \quad \dim(\operatorname{im} T) \leq \dim V$$

Indeed, $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$ by Theorem 7.2.4. Of course, the first inequality also follows because $\ker T$ is a subspace of V.

Example 7.2.9

Let $D : \mathbf{P}_n \to \mathbf{P}_{n-1}$ be the differentiation map defined by $D[p(x)] = p'(x)$. Compute $\ker D$ and hence conclude that D is onto.

Solution. Because $p'(x) = 0$ means $p(x)$ is constant, we have $\dim(\ker D) = 1$. Since $\dim \mathbf{P}_n = n+1$, the dimension theorem gives

$$\dim(\operatorname{im} D) = (n+1) - \dim(\ker D) = n = \dim(\mathbf{P}_{n-1})$$

This implies that $\operatorname{im} D = \mathbf{P}_{n-1}$, so D is onto.

Of course it is not difficult to verify directly that each polynomial $q(x)$ in $\mathbf{P}_{n-1}$ is the derivative of some polynomial in $\mathbf{P}_n$ (simply integrate $q(x)$!), so the dimension theorem is not needed in this case. However, in some situations it is difficult to see directly that a linear transformation is onto, and the method used in Example 7.2.9 may be by far the easiest way to prove it. Here is another illustration.

Example 7.2.10

Given a in $\mathbb{R}$, the evaluation map $E_a : \mathbf{P}_n \to \mathbb{R}$ is given by $E_a[p(x)] = p(a)$. Show that E_a is linear and onto, and hence conclude that $\{(x-a), (x-a)^2, \dots, (x-a)^n\}$ is a basis of $\ker E_a$, the subspace of all polynomials $p(x)$ for which $p(a) = 0$.

Solution. E_a is linear by Example 7.1.3; the verification that it is onto is left to the reader. Hence $\dim(\operatorname{im} E_a) = \dim(\mathbb{R}) = 1$, so $\dim(\ker E_a) = (n+1) - 1 = n$ by the dimension theorem. Now each of the n polynomials $(x-a), (x-a)^2, \dots, (x-a)^n$ clearly lies in $\ker E_a$, and they are linearly independent (they have distinct degrees). Hence they are a basis because $\dim(\ker E_a) = n$.

We conclude by applying the dimension theorem to the rank of a matrix.

Example 7.2.11

If A is any $m \times n$ matrix, show that $\operatorname{rank} A = \operatorname{rank} A^TA = \operatorname{rank} AA^T$.

Solution. It suffices to show that $\operatorname{rank} A = \operatorname{rank} A^TA$ (the rest follows by replacing A with A^T). Write $B = A^TA$, and consider the associated matrix transformations

$$T_A : \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad T_B : \mathbb{R}^n \to \mathbb{R}^n$$

The dimension theorem and Example 7.2.2 give

$$\begin{aligned}\text{rank } A &= \text{rank } T_A = \dim(\text{im } T_A) = n - \dim(\ker T_A)\\ \text{rank } B &= \text{rank } T_B = \dim(\text{im } T_B) = n - \dim(\ker T_B)\end{aligned}$$

so it suffices to show that $\ker T_A = \ker T_B$. Now $A\mathbf{x} = \mathbf{0}$ implies that $B\mathbf{x} = A^TA\mathbf{x} = \mathbf{0}$, so $\ker T_A$ is contained in $\ker T_B$. On the other hand, if $B\mathbf{x} = \mathbf{0}$, then $A^TA\mathbf{x} = \mathbf{0}$, so

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$$

This implies that $A\mathbf{x} = \mathbf{0}$, so $\ker T_B$ is contained in $\ker T_A$.

Exercises for 7.2

Exercise 7.2.1 For each matrix A, find a basis for the kernel and image of T_A, and find the rank and nullity of T_A.

a. $\left[\begin{array}{rrrr} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{array}\right]$ b. $\left[\begin{array}{rrrr} 2 & 1 & -1 & 3 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & -4 & 2 \end{array}\right]$

c. $\left[\begin{array}{rrr} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & -1 & 5 \\ 0 & 2 & -2 \end{array}\right]$ d. $\left[\begin{array}{rrr} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 1 & 2 & -3 \\ 0 & 3 & -6 \end{array}\right]$

Exercise 7.2.2 In each case, (i) find a basis of $\ker T$, and (ii) find a basis of $\text{im } T$. You may assume that T is linear.

a. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(a + bx + cx^2) = (a, b)$

b. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(p(x)) = (p(0), p(1))$

c. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x + y, x + y, 0)$

d. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x, x, y, y)$

e. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} a+b & b+c \\ c+d & d+a \end{array}\right]$

f. $T : \mathbf{M}_{22} \to \mathbb{R}$; $T\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = a + d$

g. $T : \mathbf{P}_n \to \mathbb{R}$; $T(r_0 + r_1x + \cdots + r_nx^n) = r_n$

h. $T : \mathbb{R}^n \to \mathbb{R}$; $T(r_1, r_2, \ldots, r_n) = r_1 + r_2 + \cdots + r_n$

i. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA - AX$, where $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$

j. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA$, where $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$

Exercise 7.2.3 Let $P : V \to \mathbb{R}$ and $Q : V \to \mathbb{R}$ be linear transformations, where V is a vector space. Define $T : V \to \mathbb{R}^2$ by $T(\mathbf{v}) = (P(\mathbf{v}), Q(\mathbf{v}))$.

a. Show that T is a linear transformation.

b. Show that $\ker T = \ker P \cap \ker Q$, the set of vectors in both $\ker P$ and $\ker Q$.

Exercise 7.2.4 In each case, find a basis $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ of V such that $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ is a basis of $\ker T$, and verify Theorem 7.2.5.

a. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x - y + 2z, x + y - z, 2x + z, 2y - 3z)$

b. $T : \mathbb{R}^3 \to \mathbb{R}^4$; $T(x, y, z) = (x + y + z, 2x - y + 3z, z - 3y, 3x + 4z)$

Exercise 7.2.5 Show that every matrix X in $\mathbf{M}_{nn}$ has the form $X = A^T - 2A$ for some matrix A in $\mathbf{M}_{nn}$. [*Hint*: The dimension theorem.]

Exercise 7.2.6 In each case either prove the statement or give an example in which it is false. Throughout, let $T : V \to W$ be a linear transformation where V and W are finite dimensional.

a. If $V = W$, then $\ker T \subseteq \operatorname{im} T$.

b. If $\dim V = 5$, $\dim W = 3$, and $\dim(\ker T) = 2$, then T is onto.

c. If $\dim V = 5$ and $\dim W = 4$, then $\ker T \neq \{\mathbf{0}\}$.

d. If $\ker T = V$, then $W = \{\mathbf{0}\}$.

e. If $W = \{\mathbf{0}\}$, then $\ker T = V$.

f. If $W = V$, and $\operatorname{im} T \subseteq \ker T$, then $T = 0$.

g. If $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of V and $T(\mathbf{e}_1) = \mathbf{0} = T(\mathbf{e}_2)$, then $\dim(\operatorname{im} T) \leq 1$.

h. If $\dim(\ker T) \leq \dim W$, then $\dim W \geq \frac{1}{2} \dim V$.

i. If T is one-to-one, then $\dim V \leq \dim W$.

j. If $\dim V \leq \dim W$, then T is one-to-one.

k. If T is onto, then $\dim V \geq \dim W$.

l. If $\dim V \geq \dim W$, then T is onto.

m. If $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is independent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is independent.

n. If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ spans V, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ spans W.

Exercise 7.2.7 Show that linear independence is preserved by one-to-one transformations and that spanning sets are preserved by onto transformations. More precisely, if $T : V \to W$ is a linear transformation, show that:

a. If T is one-to-one and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is independent in V, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is independent in W.

b. If T is onto and $V = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then $W = \operatorname{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$.

Exercise 7.2.8 Given $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in a vector space V, define $T : \mathbb{R}^n \to V$ by $T(r_1, \dots, r_n) = r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n$. Show that T is linear, and that:

a. T is one-to-one if and only if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is independent.

b. T is onto if and only if $V = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$.

Exercise 7.2.9 Let $T : V \to V$ be a linear transformation where V is finite dimensional. Show that exactly one of (i) and (ii) holds: (i) $T(\mathbf{v}) = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$ in V; (ii) $T(\mathbf{x}) = \mathbf{v}$ has a solution $\mathbf{x}$ in V for every $\mathbf{v}$ in V.

Exercise 7.2.10 Let $T : \mathbf{M}_{nn} \to \mathbb{R}$ denote the trace map: $T(A) = \operatorname{tr} A$ for all A in $\mathbf{M}_{nn}$. Show that $\dim(\ker T) = n^2 - 1$.

Exercise 7.2.11 Show that the following are equivalent for a linear transformation $T : V \to W$.

1. $\ker T = V$
2. $\operatorname{im} T = \{\mathbf{0}\}$
3. $T = 0$

Exercise 7.2.12 Let A and B be $m \times n$ and $k \times n$ matrices, respectively. Assume that $A\mathbf{x} = \mathbf{0}$ implies $B\mathbf{x} = \mathbf{0}$ for every n-column $\mathbf{x}$. Show that $\operatorname{rank} A \geq \operatorname{rank} B$. [*Hint*: Theorem 7.2.4.]

Exercise 7.2.13 Let A be an $m \times n$ matrix of rank r. Thinking of $\mathbb{R}^n$ as rows, define $V = \{\mathbf{x} \text{ in } \mathbb{R}^m \mid \mathbf{x}A = \mathbf{0}\}$. Show that $\dim V = m - r$.

Exercise 7.2.14 Consider

$$V = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \middle| a + c = b + d \right\}$$

a. Consider $S : \mathbf{M}_{22} \to \mathbb{R}$ with $S \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + c - b - d$. Show that S is linear and onto and that V is a subspace of $\mathbf{M}_{22}$. Compute $\dim V$.

b. Consider $T : V \to \mathbb{R}$ with $T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + c$. Show that T is linear and onto, and use this information to compute $\dim(\ker T)$.

Exercise 7.2.15 Define $T : \mathbf{P}_n \to \mathbb{R}$ by $T[p(x)] =$ the sum of all the coefficients of $p(x)$.

a. Use the dimension theorem to show that $\dim(\ker T) = n$.

b. Conclude that $\{x-1, x^2-1, \dots, x^n-1\}$ is a basis of ker T.

Exercise 7.2.16 Use the dimension theorem to prove Theorem 1.3.1: If A is an $m \times n$ matrix with $m < n$, the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous equations in n variables always has a nontrivial solution.

Exercise 7.2.17 Let B be an $n \times n$ matrix, and consider the subspaces $U = \{A \mid A \text{ in } \mathbf{M}_{mn},\ AB = 0\}$ and $V = \{AB \mid A \text{ in } \mathbf{M}_{mn}\}$. Show that $\dim U + \dim V = mn$.

Exercise 7.2.18 Let U and V denote, respectively, the spaces of even and odd polynomials in $\mathbf{P}_n$. Show that $\dim U + \dim V = n+1$. [*Hint*: Consider $T : \mathbf{P}_n \to \mathbf{P}_n$ where $T[p(x)] = p(x) - p(-x)$.]

Exercise 7.2.19 Show that every polynomial $f(x)$ in $\mathbf{P}_{n-1}$ can be written as $f(x) = p(x+1) - p(x)$ for some polynomial $p(x)$ in $\mathbf{P}_n$. [*Hint*: Define $T : \mathbf{P}_n \to \mathbf{P}_{n-1}$ by $T[p(x)] = p(x+1) - p(x)$.]

Exercise 7.2.20 Let U and V denote the spaces of symmetric and skew-symmetric $n \times n$ matrices. Show that $\dim U + \dim V = n^2$.

Exercise 7.2.21 Assume that B in $\mathbf{M}_{nn}$ satisfies $B^k = 0$ for some $k \geq 1$. Show that every matrix in $\mathbf{M}_{nn}$ has the form $BA - A$ for some A in $\mathbf{M}_{nn}$. [*Hint*: Show that $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ is linear and one-to-one where $T(A) = BA - A$ for each A.]

Exercise 7.2.22 Fix a column $\mathbf{y} \neq \mathbf{0}$ in $\mathbb{R}^n$ and let $U = \{A \text{ in } \mathbf{M}_{nn} \mid A\mathbf{y} = \mathbf{0}\}$. Show that $\dim U = n(n-1)$.

Exercise 7.2.23 If B in $\mathbf{M}_{mn}$ has rank r, let $U = \{A \text{ in } \mathbf{M}_{nn} \mid BA = 0\}$ and $W = \{BA \mid A \text{ in } \mathbf{M}_{nn}\}$. Show that $\dim U = n(n-r)$ and $\dim W = nr$. [*Hint*: Show that U consists of all matrices A whose columns are in the null space of B. Use Example 7.2.7.]

Exercise 7.2.24 Let $T : V \to V$ be a linear transformation where $\dim V = n$. If $\ker T \cap \operatorname{im} T = \{\mathbf{0}\}$, show that every vector $\mathbf{v}$ in V can be written $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u}$ in ker T and $\mathbf{w}$ in im T. [*Hint*: Choose bases $B \subseteq \ker T$ and $D \subseteq \operatorname{im} T$, and use Exercise 6.3.33.]

Exercise 7.2.25 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator of rank 1, where $\mathbb{R}^n$ is written as rows. Show that there exist numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that $T(X) = XA$ for all rows X in $\mathbb{R}^n$, where

$$A = \left[\begin{array}{cccc} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{array}\right]$$

[*Hint*: $\operatorname{im} T = \mathbb{R}\mathbf{w}$ for $\mathbf{w} = (b_1, \dots, b_n)$ in $\mathbb{R}^n$.]

Exercise 7.2.26 Prove Theorem 7.2.5.

Exercise 7.2.27 Let $T : V \to \mathbb{R}$ be a nonzero linear transformation, where $\dim V = n$. Show that there is a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V so that $T(r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + \cdots + r_n\mathbf{e}_n) = r_1$.

Exercise 7.2.28 Let $f \neq 0$ be a fixed polynomial of degree $m \geq 1$. If p is any polynomial, recall that $(p \circ f)(x) = p[f(x)]$. Define $T_f : P_n \to P_{n+m}$ by $T_f(p) = p \circ f$.

a. Show that T_f is linear.

b. Show that T_f is one-to-one.

Exercise 7.2.29 Let U be a subspace of a finite dimensional vector space V.

a. Show that $U = \ker T$ for some linear operator $T : V \to V$.

b. Show that $U = \operatorname{im} S$ for some linear operator $S : V \to V$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

Exercise 7.2.30 Let V and W be finite dimensional vector spaces.

a. Show that $\dim W \leq \dim V$ if and only if there exists an onto linear transformation $T : V \to W$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

b. Show that $\dim W \geq \dim V$ if and only if there exists a one-to-one linear transformation $T : V \to W$. [*Hint*: Theorem 6.4.1 and Theorem 7.1.3.]

Exercise 7.2.31 Let A and B be $n \times n$ matrices, and assume that $AXB = 0$, $X \in \mathbf{M}_{nn}$, implies $X = 0$. Show that A and B are both invertible. [Hint: Dimension Theorem.]

7.3 Isomorphisms and Composition

Often two vector spaces can consist of quite different types of vectors but, on closer examination, turn out to be the same underlying space displayed in different symbols. For example, consider the spaces

$$\mathbb{R}^2 = \{(a,\ b) \mid a,\ b \in \mathbb{R}\} \quad \text{and} \quad \mathbf{P}_1 = \{a + bx \mid a,\ b \in \mathbb{R}\}$$

Compare the addition and scalar multiplication in these spaces:

$$\begin{aligned} (a,\ b) + (a_1,\ b_1) &= (a + a_1,\ b + b_1) & \qquad (a + bx) + (a_1 + b_1x) &= (a + a_1) + (b + b_1)x \\ r(a,\ b) &= (ra,\ rb) & r(a + bx) &= (ra) + (rb)x \end{aligned}$$

Clearly these are the *same* vector space expressed in different notation: if we change each $(a,\ b)$ in $\mathbb{R}^2$ to $a + bx$, then $\mathbb{R}^2$ *becomes* $\mathbf{P}_1$, complete with addition and scalar multiplication. This can be expressed by noting that the map $(a,\ b) \mapsto a + bx$ is a linear transformation $\mathbb{R}^2 \to \mathbf{P}_1$ that is both one-to-one and onto. In this form, we can describe the general situation.

Definition 7.4 Isomorphic Vector Spaces

A linear transformation $T : V \to W$ is called an ***isomorphism*** *if it is both onto and one-to-one. The vector spaces V and W are said to be* ***isomorphic*** *if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.*

Example 7.3.1

The identity transformation $1_V : V \to V$ is an isomorphism for any vector space V.

Example 7.3.2

If $T : \mathbf{M}_{mn} \to \mathbf{M}_{nm}$ is defined by $T(A) = A^T$ for all A in $\mathbf{M}_{mn}$, then T is an isomorphism (verify). Hence $\mathbf{M}_{mn} \cong \mathbf{M}_{nm}$.

Example 7.3.3

Isomorphic spaces can "look" quite different. For example, $\mathbf{M}_{22} \cong \mathbf{P}_3$ because the map $T : \mathbf{M}_{22} \to \mathbf{P}_3$ given by $T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = a + bx + cx^2 + dx^3$ is an isomorphism (verify).

The word *isomorphism* comes from two Greek roots: *iso*, meaning "same," and *morphos*, meaning "form." An isomorphism $T : V \to W$ induces a pairing

$$\mathbf{v} \leftrightarrow T(\mathbf{v})$$