7.3 Isomorphisms and Composition

Often two vector spaces can consist of quite different types of vectors but, on closer examination, turn out to be the same underlying space displayed in different symbols. For example, consider the spaces

$$\mathbb{R}^2 = \{(a,\, b) \mid a,\, b \in \mathbb{R}\} \quad \text{and} \quad \mathbf{P}_1 = \{a + bx \mid a,\, b \in \mathbb{R}\}$$

Compare the addition and scalar multiplication in these spaces:

$$(a,\, b) + (a_1,\, b_1) = (a + a_1,\, b + b_1) \qquad (a + bx) + (a_1 + b_1 x) = (a + a_1) + (b + b_1)x$$
$$r(a,\, b) = (ra,\, rb) \qquad r(a + bx) = (ra) + (rb)x$$

Clearly these are the *same* vector space expressed in different notation: if we change each $(a,\, b)$ in $\mathbb{R}^2$ to $a + bx$, then $\mathbb{R}^2$ *becomes* $\mathbf{P}_1$, complete with addition and scalar multiplication. This can be expressed by noting that the map $(a,\, b) \mapsto a + bx$ is a linear transformation $\mathbb{R}^2 \to \mathbf{P}_1$ that is both one-to-one and onto. In this form, we can describe the general situation.

Definition 7.4 Isomorphic Vector Spaces

*A linear transformation $T : V \to W$ is called an **isomorphism** if it is both onto and one-to-one. The vector spaces V and W are said to be **isomorphic** if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.*

Example 7.3.1

The identity transformation $1_V : V \to V$ is an isomorphism for any vector space V.

Example 7.3.2

If $T : \mathbf{M}_{mn} \to \mathbf{M}_{nm}$ is defined by $T(A) = A^T$ for all A in $\mathbf{M}_{mn}$, then T is an isomorphism (verify). Hence $\mathbf{M}_{mn} \cong \mathbf{M}_{nm}$.

Example 7.3.3

Isomorphic spaces can "look" quite different. For example, $\mathbf{M}_{22} \cong \mathbf{P}_3$ because the map
$T : \mathbf{M}_{22} \to \mathbf{P}_3$ given by $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + bx + cx^2 + dx^3$ is an isomorphism (verify).

The word *isomorphism* comes from two Greek roots: *iso*, meaning "same," and *morphos*, meaning "form." An isomorphism $T : V \to W$ induces a pairing

$$\mathbf{v} \leftrightarrow T(\mathbf{v})$$

between vectors $\mathbf{v}$ in V and vectors $T(\mathbf{v})$ in W that preserves vector addition and scalar multiplication. Hence, *as far as their vector space properties are concerned*, the spaces V and W are identical except for notation. Because addition and scalar multiplication in either space are completely determined by the same operations in the other space, all *vector space* properties of either space are completely determined by those of the other.

One of the most important examples of isomorphic spaces was considered in Chapter 4. Let A denote the set of all "arrows" with tail at the origin in space, and make A into a vector space using the parallelogram law and the scalar multiple law (see Section 4.1). Then define a transformation $T : \mathbb{R}^3 \to A$ by taking

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{the arrow } \mathbf{v} \text{ from the origin to the point } P(x,\ y,\ z).$$

In Section 4.1 matrix addition and scalar multiplication were shown to correspond to the parallelogram law and the scalar multiplication law for these arrows, so the map T is a linear transformation. Moreover T is an isomorphism: it is one-to-one by Theorem 4.1.2, and it is onto because, given an arrow $\mathbf{v}$ in A with tip $P(x,\ y,\ z)$, we have $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{v}$. This justifies the identification $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in Chapter 4 of the geometric arrows with the algebraic matrices. This identification is very useful. The arrows give a "picture" of the matrices and so bring geometric intuition into $\mathbb{R}^3$; the matrices are useful for detailed calculations and so bring analytic precision into geometry. This is one of the best examples of the power of an isomorphism to shed light on both spaces being considered.

The following theorem gives a very useful characterization of isomorphisms: They are the linear transformations that preserve bases.

> **Theorem 7.3.1**
>
> *If V and W are finite dimensional spaces, the following conditions are equivalent for a linear transformation $T : V \to W$.*
>
> 1. *T is an isomorphism.*
>
> 2. *If $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$ is any basis of V, then $\{T(\mathbf{e}_1),\ T(\mathbf{e}_2),\ \ldots,\ T(\mathbf{e}_n)\}$ is a basis of W.*
>
> 3. *There exists a basis $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$ of V such that $\{T(\mathbf{e}_1),\ T(\mathbf{e}_2),\ \ldots,\ T(\mathbf{e}_n)\}$ is a basis of W.*

Proof. $(1) \Rightarrow (2)$. Let $\{\mathbf{e}_1,\ \ldots,\ \mathbf{e}_n\}$ be a basis of V. If $t_1 T(\mathbf{e}_1) + \cdots + t_n T(\mathbf{e}_n) = \mathbf{0}$ with t_i in $\mathbb{R}$, then $T(t_1 \mathbf{e}_1 + \cdots + t_n \mathbf{e}_n) = \mathbf{0}$, so $t_1 \mathbf{e}_1 + \cdots + t_n \mathbf{e}_n = \mathbf{0}$ (because $\ker T = \{\mathbf{0}\}$). But then each $t_i = 0$ by the independence of the $\mathbf{e}_i$, so $\{T(\mathbf{e}_1),\ \ldots,\ T(\mathbf{e}_n)\}$ is independent. To show that it spans W, choose $\mathbf{w}$ in W. Because T is onto, $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V, so write $\mathbf{v} = t_1 \mathbf{e}_1 + \cdots + t_n \mathbf{e}_n$. Hence we obtain $\mathbf{w} = T(\mathbf{v}) = t_1 T(\mathbf{e}_1) + \cdots + t_n T(\mathbf{e}_n)$, proving that $\{T(\mathbf{e}_1),\ \ldots,\ T(\mathbf{e}_n)\}$ spans W.

$(2) \Rightarrow (3)$. This is because V has a basis.

$(3) \Rightarrow (1)$. If $T(\mathbf{v}) = \mathbf{0}$, write $\mathbf{v} = v_1 \mathbf{e}_1 + \cdots + v_n \mathbf{e}_n$ where each v_i is in $\mathbb{R}$. Then

$$\mathbf{0} = T(\mathbf{v}) = v_1 T(\mathbf{e}_1) + \cdots + v_n T(\mathbf{e}_n)$$

so $v_1 = \cdots = v_n = 0$ by (3). Hence $\mathbf{v} = \mathbf{0}$, so $\ker T = \{\mathbf{0}\}$ and T is one-to-one. To show that T is onto, let $\mathbf{w}$ be any vector in W. By (3) there exist $w_1, \ldots, w_n$ in $\mathbb{R}$ such that

$$\mathbf{w} = w_1 T(\mathbf{e}_1) + \cdots + w_n T(\mathbf{e}_n) = T(w_1 \mathbf{e}_1 + \cdots + w_n \mathbf{e}_n)$$

Thus T is onto. □

Theorem 7.3.1 dovetails nicely with Theorem 7.1.3 as follows. Let V and W be vector spaces of dimension n, and suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ are bases of V and W, respectively. Theorem 7.1.3 asserts that there exists a linear transformation $T : V \to W$ such that

$$T(\mathbf{e}_i) = \mathbf{f}_i \quad \text{for each } i = 1, 2, \ldots, n$$

Then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is evidently a basis of W, so T is an isomorphism by Theorem 7.3.1. Furthermore, the action of T is prescribed by

$$T(r_1 \mathbf{e}_1 + \cdots + r_n \mathbf{e}_n) = r_1 \mathbf{f}_1 + \cdots + r_n \mathbf{f}_n$$

so isomorphisms between spaces of equal dimension can be easily defined as soon as bases are known. In particular, this shows that if two vector spaces V and W have the same dimension then they are isomorphic, that is $V \cong W$. This is half of the following theorem.

Theorem 7.3.2

If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Proof. It remains to show that if $V \cong W$ then $\dim V = \dim W$. But if $V \cong W$, then there exists an isomorphism $T : V \to W$. Since V is finite dimensional, let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of V. Then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is a basis of W by Theorem 7.3.1, so $\dim W = n = \dim V$. □

Corollary 7.3.1

Let U, V, and W denote vector spaces. Then:

1. $V \cong V$ for every vector space V.

2. If $V \cong W$ then $W \cong V$.

3. If $U \cong V$ and $V \cong W$, then $U \cong W$.

The proof is left to the reader. By virtue of these properties, the relation $\cong$ is called an *equivalence relation* on the class of finite dimensional vector spaces. Since $\dim(\mathbb{R}^n) = n$ it follows that

---

Corollary 7.3.2

If V is a vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

If V is a vector space of dimension n, note that there are important explicit isomorphisms $V \to \mathbb{R}^n$. Fix a basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of V and write $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for the standard basis of $\mathbb{R}^n$. By Theorem 7.1.3 there is a unique linear transformation $C_B : V \to \mathbb{R}^n$ given by

$$C_B(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where each v_i is in $\mathbb{R}$. Moreover, $C_B(\mathbf{b}_i) = \mathbf{e}_i$ for each i so C_B is an isomorphism by Theorem 7.3.1, called the **coordinate isomorphism** corresponding to the basis B. These isomorphisms will play a central role in Chapter 9.

The conclusion in the above corollary can be phrased as follows: As far as vector space properties are concerned, every n-dimensional vector space V is essentially the same as $\mathbb{R}^n$; they are the "same" vector space except for a change of symbols. This appears to make the process of abstraction seem less important—just study $\mathbb{R}^n$ and be done with it! But consider the different "feel" of the spaces $\mathbf{P}_8$ and $\mathbf{M}_{33}$ even though they are both the "same" as $\mathbb{R}^9$: For example, vectors in $\mathbf{P}_8$ can have roots, while vectors in $\mathbf{M}_{33}$ can be multiplied. So the merit in the abstraction process lies in identifying *common* properties of the vector spaces in the various examples. This is important even for finite dimensional spaces. However, the payoff from abstraction is much greater in the infinite dimensional case, particularly for spaces of functions.

Example 7.3.4

Let V denote the space of all 2×2 symmetric matrices. Find an isomorphism $T : \mathbf{P}_2 \to V$ such that $T(1) = I$, where I is the 2×2 identity matrix.

Solution. $\{1, x, x^2\}$ is a basis of $\mathbf{P}_2$, and we want a basis of V containing I. The set
$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$ is independent in V, so it is a basis because $\dim V = 3$ (by Example 6.3.11). Hence define $T : \mathbf{P}_2 \to V$ by taking $T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $T(x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and extending linearly as in Theorem 7.1.3. Then T is an isomorphism by Theorem 7.3.1, and its action is given by

$$T(a + bx + cx^2) = aT(1) + bT(x) + cT(x^2) = \begin{bmatrix} a & b \\ b & a+c \end{bmatrix}$$

The dimension theorem (Theorem 7.2.4) gives the following useful fact about isomorphisms.

Theorem 7.3.3

If V and W have the same dimension n, a linear transformation $T : V \to W$ is an isomorphism if it is either one-to-one or onto.

Proof. The dimension theorem asserts that $\dim(\ker T) + \dim(\operatorname{im} T) = n$, so $\dim(\ker T) = 0$ if and only if $\dim(\operatorname{im} T) = n$. Thus T is one-to-one if and only if T is onto, and the result follows. $\quad\square$

Composition

Suppose that $T : V \to W$ and $S : W \to U$ are linear transformations. They link together as in the diagram so, as in Section 2.3, it is possible to define a new function $V \to U$ by first applying T and then S.

Definition 7.5 Composition of Linear Transformations
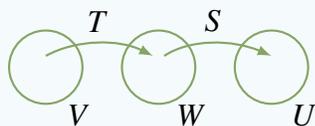
Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$, the **composite** $ST : V \to U$ of T and S is defined by

$$ST(\mathbf{v}) = S[T(\mathbf{v})] \quad \text{for all } \mathbf{v} \text{ in } V$$

The operation of forming the new function ST is called **composition**.[1]

The action of ST can be described compactly as follows: ST means first T then S.

Not all pairs of linear transformations can be composed. For example, if $T : V \to W$ and $S : W \to U$ are linear transformations then $ST : V \to U$ is defined, but TS *cannot* be formed unless $U = V$ because $S : W \to U$ and $T : V \to W$ do not "link" in that order.[2]

Moreover, even if ST and TS can both be formed, they may not be equal. In fact, if $S : \mathbb{R}^m \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are induced by matrices A and B respectively, then ST and TS can both be formed (they are induced by AB and BA respectively), but the matrix products AB and BA may not be equal (they may not even be the same *size*). Here is another example.

Example 7.3.5

Define: $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ and $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ by $S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $T(A) = A^T$ for $A \in \mathbf{M}_{22}$. Describe the action of ST and TS, and show that $ST \neq TS$.

Solution. $ST\begin{bmatrix} a & b \\ c & d \end{bmatrix} = S\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$, whereas

$TS\begin{bmatrix} a & b \\ c & d \end{bmatrix} = T\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$.

It is clear that $TS\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ need not equal $ST\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $TS \neq ST$.

The next theorem collects some basic properties of the composition operation.

---

[1] In Section 2.3 we denoted the composite as $S \circ T$. However, it is more convenient to use the simpler notation ST.
[2] Actually, all that is required is $U \subseteq V$.

> **Theorem 7.3.4:** [3]
>
> Let $V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$ be linear transformations.
>
> 1. The composite ST is again a linear transformation.
>
> 2. $T1_V = T$ and $1_W T = T$.
>
> 3. $(RS)T = R(ST)$.

Proof. The proofs of (1) and (2) are left as Exercise 7.3.25. To prove (3), observe that, for all $\mathbf{v}$ in V:

$$\{(RS)T\}(\mathbf{v}) = (RS)[T(\mathbf{v})] = R\{S[T(\mathbf{v})]\} = R\{(ST)(\mathbf{v})\} = \{R(ST)\}(\mathbf{v})$$

□

    Up to this point, composition seems to have no connection with isomorphisms. In fact, the two notions are closely related.

> **Theorem 7.3.5**
>
> Let V and W be finite dimensional vector spaces. The following conditions are equivalent for a linear transformation $T : V \to W$.
>
> 1. T is an isomorphism.
>
> 2. There exists a linear transformation $S : W \to V$ such that $ST = 1_V$ and $TS = 1_W$.
>
> Moreover, in this case S is also an isomorphism and is uniquely determined by T:
>
> $$\text{If } \mathbf{w} \text{ in } W \text{ is written as } \mathbf{w} = T(\mathbf{v}), \text{ then } S(\mathbf{w}) = \mathbf{v}.$$

Proof. (1) $\Rightarrow$ (2). If $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V, then $D = \{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)\}$ is a basis of W by Theorem 7.3.1. Hence (using Theorem 7.1.3), define a linear transformation $S : W \to V$ by

$$S[T(\mathbf{e}_i)] = \mathbf{e}_i \quad \text{for each } i \tag{7.2}$$

Since $\mathbf{e}_i = 1_V(\mathbf{e}_i)$, this gives $ST = 1_V$ by Theorem 7.1.2. But applying T gives $T[S[T(\mathbf{e}_i)]] = T(\mathbf{e}_i)$ for each i, so $TS = 1_W$ (again by Theorem 7.1.2, using the basis D of W).

    (2) $\Rightarrow$ (1). If $T(\mathbf{v}) = T(\mathbf{v}_1)$, then $S[T(\mathbf{v})] = S[T(\mathbf{v}_1)]$. Because $ST = 1_V$ by (2), this reads $\mathbf{v} = \mathbf{v}_1$; that is, T is one-to-one. Given $\mathbf{w}$ in W, the fact that $TS = 1_W$ means that $\mathbf{w} = T[S(\mathbf{w})]$, so T is onto.

---

[3]Theorem 7.3.4 can be expressed by saying that vector spaces and linear transformations are an example of a *category*. In general a category consists of certain *objects* and, for any two objects X and Y, a set $\text{mor}(X, Y)$. The elements α of $\text{mor}(X, Y)$ are called *morphisms* from X to Y and are written $\alpha : X \to Y$. It is assumed that identity morphisms and composition are defined in such a way that Theorem 7.3.4 holds. Hence, in the category of vector spaces the objects are the vector spaces themselves and the morphisms are the linear transformations. Another example is the category of metric spaces, in which the objects are sets equipped with a distance function (called a metric), and the morphisms are continuous functions (with respect to the metric). The category of sets and functions is a very basic example.

Finally, S is uniquely determined by the condition $ST = 1_V$ because this condition implies (7.2). S is an isomorphism because it carries the basis D to B. As to the last assertion, given $\mathbf{w}$ in W, write $\mathbf{w} = r_1 T(\mathbf{e}_1) + \cdots + r_n T(\mathbf{e}_n)$. Then $\mathbf{w} = T(\mathbf{v})$, where $\mathbf{v} = r_1 \mathbf{e}_1 + \cdots + r_n \mathbf{e}_n$. Then $S(\mathbf{w}) = \mathbf{v}$ by (7.2).   □

Given an isomorphism $T : V \to W$, the unique isomorphism $S : W \to V$ satisfying condition (2) of Theorem 7.3.5 is called the **inverse** of T and is denoted by T^{-1}. Hence $T : V \to W$ and $T^{-1} : W \to V$ are related by the **fundamental identities**:

$$T^{-1}[T(\mathbf{v})] = \mathbf{v} \text{ for all } \mathbf{v} \text{ in } V \quad \text{and} \quad T[T^{-1}(\mathbf{w})] = \mathbf{w} \text{ for all } \mathbf{w} \text{ in } W$$

In other words, each of T and T^{-1} reverses the action of the other. In particular, equation (7.2) in the proof of Theorem 7.3.5 shows how to define T^{-1} using the image of a basis under the isomorphism T. Here is an example.

Example 7.3.6

Define $T : \mathbf{P}_1 \to \mathbf{P}_1$ by $T(a + bx) = (a - b) + ax$. Show that T has an inverse, and find the action of T^{-1}.

Solution. The transformation T is linear (verify). Because $T(1) = 1 + x$ and $T(x) = -1$, T carries the basis $B = \{1, \ x\}$ to the basis $D = \{1 + x, \ -1\}$. Hence T is an isomorphism, and T^{-1} carries D back to B, that is,
$$T^{-1}(1 + x) = 1 \quad \text{and} \quad T^{-1}(-1) = x$$
Because $a + bx = b(1 + x) + (b - a)(-1)$, we obtain
$$T^{-1}(a + bx) = bT^{-1}(1 + x) + (b - a)T^{-1}(-1) = b + (b - a)x$$

Sometimes the action of the inverse of a transformation is apparent.

Example 7.3.7

If $B = \{\mathbf{b}_1, \ \mathbf{b}_2, \ \ldots, \ \mathbf{b}_n\}$ is a basis of a vector space V, the coordinate transformation $C_B : V \to \mathbb{R}^n$ is an isomorphism defined by

$$C_B(v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + \cdots + v_n \mathbf{b}_n) = (v_1, \ v_2, \ \ldots, \ v_n)^T$$

The way to reverse the action of C_B is clear: $C_B^{-1} : \mathbb{R}^n \to V$ is given by

$$C_B^{-1}(v_1, \ v_2, \ \ldots, \ v_n) = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + \cdots + v_n \mathbf{b}_n \quad \text{for all } v_i \text{ in } V$$

Condition (2) in Theorem 7.3.5 characterizes the inverse of a linear transformation $T : V \to W$ as the (unique) transformation $S : W \to V$ that satisfies $ST = 1_V$ and $TS = 1_W$. This often determines the inverse.

Example 7.3.8

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (z, x, y)$. Show that $T^3 = 1_{\mathbb{R}^3}$, and hence find T^{-1}.

Solution. $T^2(x, y, z) = T[T(x, y, z)] = T(z, x, y) = (y, z, x)$. Hence

$$T^3(x, y, z) = T[T^2(x, y, z)] = T(y, z, x) = (x, y, z)$$

Since this holds for all (x, y, z), it shows that $T^3 = 1_{\mathbb{R}^3}$, so $T(T^2) = 1_{\mathbb{R}^3} = (T^2)T$. Thus $T^{-1} = T^2$ by (2) of Theorem 7.3.5.

Example 7.3.9

Define $T : \mathbf{P}_n \to \mathbb{R}^{n+1}$ by $T(p) = (p(0), p(1), \ldots, p(n))$ for all p in $\mathbf{P}_n$. Show that T^{-1} exists.

Solution. The verification that T is linear is left to the reader. If $T(p) = 0$, then $p(k) = 0$ for $k = 0, 1, \ldots, n$, so p has $n+1$ distinct roots. Because p has degree at most n, this implies that $p = 0$ is the zero polynomial (Theorem 6.5.4) and hence that T is one-to-one. But $\dim \mathbf{P}_n = n+1 = \dim \mathbb{R}^{n+1}$, so this means that T is also onto and hence is an isomorphism. Thus T^{-1} exists by Theorem 7.3.5. Note that we have not given a description of the action of T^{-1}, we have merely shown that such a description exists. To give it explicitly requires some ingenuity; one method involves the Lagrange interpolation expansion (Theorem 6.5.3).

Exercises for 7.3

Exercise 7.3.1 Verify that each of the following is an isomorphism (Theorem 7.3.3 is useful).

a. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x+y, y+z, z+x)$

b. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x, x+y, x+y+z)$

c. $T : \mathbb{C} \to \mathbb{C}$; $T(z) = \bar{z}$

d. $T : \mathbf{M}_{mn} \to \mathbf{M}_{mn}$; $T(X) = UXV$, U and V invertible

e. $T : \mathbf{P}_1 \to \mathbb{R}^2$; $T[p(x)] = [p(0), p(1)]$

f. $T : V \to V$; $T(\mathbf{v}) = k\mathbf{v}$, $k \neq 0$ a fixed number, V any vector space

g. $T : \mathbf{M}_{22} \to \mathbb{R}^4$; $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b, d, c, a-b)$

h. $T : \mathbf{M}_{mn} \to \mathbf{M}_{nm}$; $T(A) = A^T$

Exercise 7.3.2 Show that

$$\{a+bx+cx^2, a_1+b_1x+c_1x^2, a_2+b_2x+c_2x^2\}$$

is a basis of $\mathbf{P}_2$ if and only if

$\{(a, b, c), (a_1, b_1, c_1), (a_2, b_2, c_2)\}$ is a basis of $\mathbb{R}^3$.

Exercise 7.3.3 If V is any vector space, let V^n denote the space of all n-tuples $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$, where each $\mathbf{v}_i$ lies in V. (This is a vector space with component-wise operations; see Exercise 6.1.17.) If $C_j(A)$ denotes the jth column of the $m \times n$ matrix A, show that $T : \mathbf{M}_{mn} \to (\mathbb{R}^m)^n$ is an isomorphism if $T(A) = [\ C_1(A)\ \ C_2(A)\ \ \cdots\ \ C_n(A)\ ]$. (Here $\mathbb{R}^m$ consists of columns.)

Exercise 7.3.4 In each case, compute the action of ST and TS, and show that $ST \neq TS$.

   a. $S : \mathbb{R}^2 \to \mathbb{R}^2$ with $S(x, y) = (y, x)$; $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x, y) = (x, 0)$

   b. $S : \mathbb{R}^3 \to \mathbb{R}^3$ with $S(x, y, z) = (x, 0, z)$; $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $T(x, y, z) = (x + y, 0, y + z)$

   c. $S : \mathbf{P}_2 \to \mathbf{P}_2$ with $S(p) = p(0) + p(1)x + p(2)x^2$; $T : \mathbf{P}_2 \to \mathbf{P}_2$ with $T(a + bx + cx^2) = b + cx + ax^2$

   d. $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ with $S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$;

      $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ with $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$

Exercise 7.3.5 In each case, show that the linear transformation T satisfies $T^2 = T$.

   a. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x, y, z, w) = (x, 0, z, 0)$

   b. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x, y) = (x + y, 0)$

   c. $T : \mathbf{P}_2 \to \mathbf{P}_2$; $T(a + bx + cx^2) = (a + b - c) + cx + cx^2$

   d. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a + c & b + d \\ a + c & b + d \end{bmatrix}$

Exercise 7.3.6 Determine whether each of the following transformations T has an inverse and, if so, determine the action of T^{-1}.

   a. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x + y, y + z, z + x)$

   b. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x, y, z, t) = (x + y, y + z, z + t, t + x)$

   c. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - c & b - d \\ 2a - c & 2b - d \end{bmatrix}$

   d. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 3c - a & 3d - b \end{bmatrix}$

   e. $T : \mathbf{P}_2 \to \mathbb{R}^3$; $T(a + bx + cx^2) = (a - c, 2b, a + c)$

   f. $T : \mathbf{P}_2 \to \mathbb{R}^3$; $T(p) = [p(0), p(1), p(-1)]$

Exercise 7.3.7 In each case, show that T is self-inverse, that is: $T^{-1} = T$.

   a. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x, y, z, w) = (x, -y, -z, w)$

   b. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(x, y) = (ky - x, y)$, k any fixed number

   c. $T : \mathbf{P}_n \to \mathbf{P}_n$; $T(p(x)) = p(3 - x)$

   d. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = AX$ where $A = \frac{1}{4} \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}$

Exercise 7.3.8 In each case, show that $T^6 = 1_{R^4}$ and so determine T^{-1}.

   a. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x, y, z, w) = (-x, z, w, y)$

   b. $T : \mathbb{R}^4 \to \mathbb{R}^4$; $T(x, y, z, w) = (-y, x - y, z, -w)$

Exercise 7.3.9 In each case, show that T is an isomorphism by defining T^{-1} explicitly.

   a. $T : \mathbf{P}_n \to \mathbf{P}_n$ is given by $T[p(x)] = p(x + 1)$.

   b. $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ is given by $T(A) = UA$ where U is invertible in $\mathbf{M}_{nn}$.

Exercise 7.3.10 Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$:

   a. If S and T are both one-to-one, show that ST is one-to-one.

   b. If S and T are both onto, show that ST is onto.

Exercise 7.3.11 Let $T : V \to W$ be a linear transformation.

   a. If T is one-to-one and $TR = TR_1$ for transformations R and $R_1 : U \to V$, show that $R = R_1$.

   b. If T is onto and $ST = S_1T$ for transformations S and $S_1 : W \to U$, show that $S = S_1$.

Exercise 7.3.12 Consider the linear transformations $V \xrightarrow{T} W \xrightarrow{R} U$.

   a. Show that $\ker T \subseteq \ker RT$.

   b. Show that $\operatorname{im} RT \subseteq \operatorname{im} R$.

Exercise 7.3.13 Let $V \xrightarrow{T} U \xrightarrow{S} W$ be linear transformations.

a. If ST is one-to-one, show that T is one-to-one and that $\dim V \leq \dim U$.

b. If ST is onto, show that S is onto and that $\dim W \leq \dim U$.

Exercise 7.3.14 Let $T : V \to V$ be a linear transformation. Show that $T^2 = 1_V$ if and only if T is invertible and $T = T^{-1}$.

Exercise 7.3.15 Let N be a nilpotent $n \times n$ matrix (that is, $N^k = 0$ for some k). Show that $T : \mathbf{M}_{nm} \to \mathbf{M}_{nm}$ is an isomorphism if $T(X) = X - NX$. [*Hint*: If X is in $\ker T$, show that $X = NX = N^2X = \cdots$. Then use Theorem 7.3.3.]

Exercise 7.3.16 Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ be any basis of V such that $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ is a basis of $\ker T$. Show that $\operatorname{im} T \cong \operatorname{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$. [*Hint*: See Theorem 7.2.5.]

Exercise 7.3.17 Is every isomorphism $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ given by an invertible matrix U such that $T(X) = UX$ for all X in $\mathbf{M}_{22}$? Prove your answer.

Exercise 7.3.18 Let $\mathbf{D}_n$ denote the space of all functions f from $\{1, 2, \ldots, n\}$ to $\mathbb{R}$ (see Exercise 6.3.35). If $T : \mathbf{D}_n \to \mathbb{R}^n$ is defined by

$$T(f) = (f(1), f(2), \ldots, f(n)),$$

show that T is an isomorphism.

Exercise 7.3.19

a. Let V be the vector space of Exercise 6.1.3. Find an isomorphism $T : V \to \mathbb{R}^1$.

b. Let V be the vector space of Exercise 6.1.4. Find an isomorphism $T : V \to \mathbb{R}^2$.

Exercise 7.3.20 Let $V \xrightarrow{T} W \xrightarrow{S} V$ be linear transformations such that $ST = 1_V$. If $\dim V = \dim W = n$, show that $S = T^{-1}$ and $T = S^{-1}$. [*Hint*: Exercise 7.3.13 and Theorem 7.3.3, Theorem 7.3.4, and Theorem 7.3.5.]

Exercise 7.3.21 Let $V \xrightarrow{T} W \xrightarrow{S} V$ be functions such that $TS = 1_W$ and $ST = 1_V$. If T is linear, show that S is also linear.

Exercise 7.3.22 Let A and B be matrices of size $p \times m$ and $n \times q$. Assume that $mn = pq$. Define $R : \mathbf{M}_{mn} \to \mathbf{M}_{pq}$ by $R(X) = AXB$.

a. Show that $\mathbf{M}_{mn} \cong \mathbf{M}_{pq}$ by comparing dimensions.

b. Show that R is a linear transformation.

c. Show that if R is an isomorphism, then $m = p$ and $n = q$. [*Hint*: Show that $T : \mathbf{M}_{mn} \to \mathbf{M}_{pn}$ given by $T(X) = AX$ and $S : \mathbf{M}_{mn} \to \mathbf{M}_{mq}$ given by $S(X) = XB$ are both one-to-one, and use the dimension theorem.]

Exercise 7.3.23 Let $T : V \to V$ be a linear transformation such that $T^2 = 0$ is the zero transformation.

a. If $V \neq \{\mathbf{0}\}$, show that T cannot be invertible.

b. If $R : V \to V$ is defined by $R(\mathbf{v}) = \mathbf{v} + T(\mathbf{v})$ for all $\mathbf{v}$ in V, show that R is linear and invertible.

Exercise 7.3.24 Let V consist of all sequences $[x_0, x_1, x_2, \ldots)$ of numbers, and define vector operations

$$[x_o, x_1, \ldots) + [y_0, y_1, \ldots) = [x_0 + y_0, x_1 + y_1, \ldots)$$
$$r[x_0, x_1, \ldots) = [rx_0, rx_1, \ldots)$$

a. Show that V is a vector space of infinite dimension.

b. Define $T : V \to V$ and $S : V \to V$ by $T[x_0, x_1, \ldots) = [x_1, x_2, \ldots)$ and $S[x_0, x_1, \ldots) = [0, x_0, x_1, \ldots)$. Show that $TS = 1_V$, so TS is one-to-one and onto, but that T is not one-to-one and S is not onto.

Exercise 7.3.25 Prove (1) and (2) of Theorem 7.3.4.

Exercise 7.3.26 Define $T : \mathbf{P}_n \to \mathbf{P}_n$ by $T(p) = p(x) + xp'(x)$ for all p in $\mathbf{P}_n$.

a. Show that T is linear.

b. Show that $\ker T = \{\mathbf{0}\}$ and conclude that T is an isomorphism. [*Hint*: Write $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and compare coefficients if $p(x) = -xp'(x)$.]

c. Conclude that each $q(x)$ in $\mathbf{P}_n$ has the form $q(x) = p(x) + xp'(x)$ for some unique polynomial $p(x)$.

d. Does this remain valid if T is defined by $T[p(x)] = p(x) - xp'(x)$? Explain.

Exercise 7.3.27 Let $T : V \to W$ be a linear transformation, where V and W are finite dimensional.

a. Show that T is one-to-one if and only if there exists a linear transformation $S : W \to V$ with $ST = 1_V$. [*Hint*: If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V and T is one-to-one, show that W has a basis $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n), \mathbf{f}_{n+1}, \ldots, \mathbf{f}_{n+k}\}$ and use Theorem 7.1.2 and Theorem 7.1.3.]

b. Show that T is onto if and only if there exists a linear transformation $S : W \to V$ with $TS = 1_W$. [*Hint*: Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \ldots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ is a basis of ker T. Use Theorem 7.2.5, Theorem 7.1.2 and Theorem 7.1.3.]

Exercise 7.3.28 Let S and T be linear transformations $V \to W$, where dim $V = n$ and dim $W = m$.

a. Show that ker $S = $ ker T if and only if $T = RS$ for some isomorphism $R : W \to W$. [*Hint*: Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \ldots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ is a basis of ker $S = $ ker T. Use Theorem 7.2.5 to extend $\{S(\mathbf{e}_1), \ldots, S(\mathbf{e}_r)\}$ and $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_r)\}$ to bases of W.]

b. Show that im $S = $ im T if and only if $T = SR$ for some isomorphism $R : V \to V$. [*Hint*: Show that dim (ker S) = dim (ker T) and choose bases $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \ldots, \mathbf{f}_r, \ldots, \mathbf{f}_n\}$ of V where $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_{r+1}, \ldots, \mathbf{f}_n\}$ are bases of ker S and ker T, respectively. If $1 \leq i \leq r$, show that $S(\mathbf{e}_i) = T(\mathbf{g}_i)$ for some $\mathbf{g}_i$ in V, and prove that $\{\mathbf{g}_1, \ldots, \mathbf{g}_r, \mathbf{f}_{r+1}, \ldots, \mathbf{f}_n\}$ is a basis of V.]

Exercise 7.3.29 If $T : V \to V$ is a linear transformation where dim $V = n$, show that $TST = T$ for some isomorphism $S : V \to V$. [*Hint*: Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ be as in Theorem 7.2.5. Extend $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_r)\}$ to a basis of V, and use Theorem 7.3.1, Theorem 7.1.2 and Theorem 7.1.3.]

Exercise 7.3.30 Let A and B denote $m \times n$ matrices. In each case show that (1) and (2) are equivalent.

a. (1) A and B have the same null space. (2) $B = PA$ for some invertible $m \times m$ matrix P.

b. (1) A and B have the same range. (2) $B = AQ$ for some invertible $n \times n$ matrix Q.

[*Hint*: Use Exercise 7.3.28.]

7.4 A Theorem about Differential Equations

Differential equations are instrumental in solving a variety of problems throughout science, social science, and engineering. In this brief section, we will see that the set of solutions of a linear differential equation (with constant coefficients) is a vector space and we will calculate its dimension. The proof is pure linear algebra, although the applications are primarily in analysis. However, a key result (Lemma 7.4.3 below) can be applied much more widely.

We denote the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ by f', and f will be called **differentiable** if it can be differentiated any number of times. If f is a differentiable function, the nth derivative $f^{(n)}$ of f is the result of differentiating n times. Thus $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f^{(1)\prime}$, $\ldots$, and in general $f^{(n+1)} = f^{(n)\prime}$ for each $n \geq 0$. For small values of n these are often written as f, f', f'', f''', $\ldots$.

If a, b, and c are numbers, the differential equations

$$f'' - af' - bf = 0 \quad \text{or} \quad f''' - af'' - bf' - cf = 0$$

are said to be of **second order** and **third-order**, respectively. In general, an equation

$$f^{(n)} - a_{n-1}f^{(n-1)} - a_{n-2}f^{(n-2)} - \cdots - a_2 f^{(2)} - a_1 f^{(1)} - a_0 f^{(0)} = 0, \ a_i \text{ in } \mathbb{R} \tag{7.3}$$