Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations
§1-1. Solutions and Elementary Operations

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Emory University, 2020 Fall

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Solutions of Linear Equations

Elementary Operations

The Augmented Matrix

Solving a System using Back Substitution

Solutions of Linear Equations

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Example

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Objective:

Can we do the same for linear equations in more variables?

Definition

A linear equation is an expression

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $n \geq 1$, $a_1, \ldots, a_n$ are real numbers, not all of them equal to zero, and b is a real number.

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A solution to a system of m equations in n variables is an n-tuple of numbers that satisfy each of the equations.

Solve a system means 'find all solutions to the system'.

Example

A system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0 \end{array}$$

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- coefficients:

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- constant terms:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & {\color{red}-1} \\ -x_1 & + & 3x_2 & + & 6x_3 & = & {\color{red}0} \end{array}$$

Example (continued)

$x_1 = -3$, $x_2 = -1$, $x_3 = 0$ is a solution to the system

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However, $x_1 = -1$, $x_2 = 0$, $x_3 = 0$ is not a solution to the system, because

$$\begin{array}{rcrcrcrl} (-1) & - & 2 \cdot 0 & - & 7 \cdot 0 & = & -1 & \\ -(-1) & + & 3 \cdot 0 & + & 6 \cdot 0 & = & 1 & \neq 0 \end{array}$$

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The system above is consistent, meaning that the system has at least one solution.

Example (continued)

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & -8 \end{array}$$

is an example of an inconsistent system, meaning that it has no solutions.

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Why are there no solutions?

Example

Consider the system of linear equations in two variables

$$\begin{cases} \mathrm{x} + \mathrm{y} = 3 \\ \mathrm{y} - \mathrm{x} = 5 \end{cases}$$

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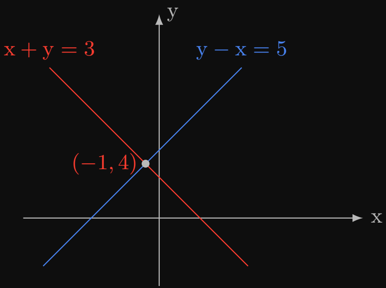

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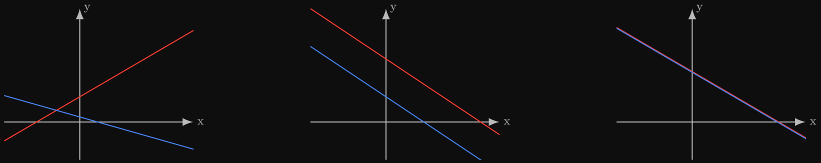

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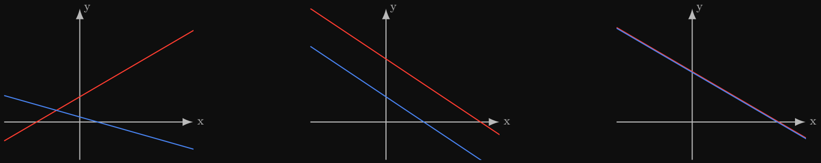

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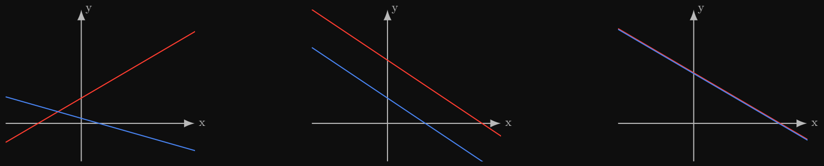

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(unique solution)

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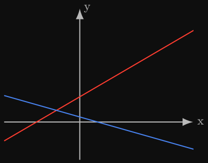

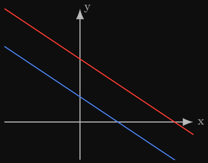

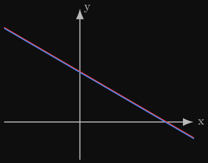

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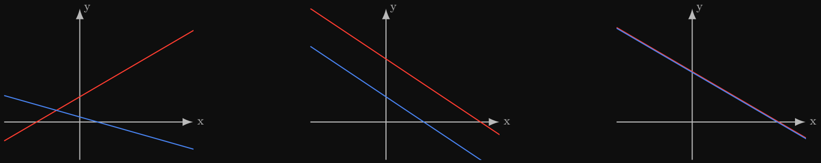

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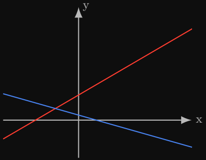

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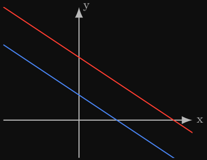

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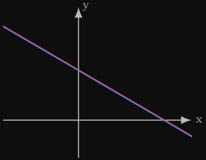

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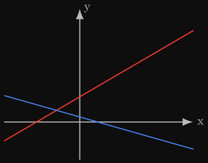

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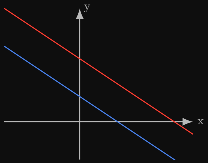

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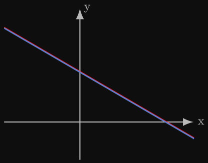

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Remark

We will see in what follows that this generalizes to systems of linear equations in more than two variables.

Example

The system of linear equations in three variables that we saw earlier

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 0, \end{array}$$

has solutions $x_1 = -3 + 9s$, $x_2 = -1 + s$, $x_3 = s$ where s is any real number (written $s \in \mathbb{R}$).

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s is called a parameter, and the expression

$$x_1 = -3 + 9s, \quad x_2 = -1 + s, \quad x_3 = s, \quad \text{where } s \in \mathbb{R}$$

is called the general solution in parametric form.

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Example

The two systems of linear equations

$$\begin{array}{rcrcr} 2x & + & y & = & 2 \\ 3x & & & = & 3 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcr} x & + & y & = & 1 \\ & & y & = & 0 \end{array}$$

are equivalent because both systems have the unique solution $x = 1$, $y = 0$.

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- Type I: Interchange two equations, $r_1 \leftrightarrow r_2$.
- Type II: Multiply an equation by a nonzero number, $-2r_1$.
- Type III: Add a multiple of one equation to a different equation, $3r_3 + r_2$.

Example

Consider the system of linear eq's:
$$\begin{array}{rcrcrcr} 3x_1 & - & 2x_2 & - & 7x_3 & = & -1 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

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1. Interchange first two equations (Type I):

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2. Multiply first equation by -2 (Type II):

$$-2r_1 \qquad \begin{array}{rcrcrcr} -6x_1 & + & 4x_2 & + & 14x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

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3. Add 3 time the second equation to the first equation (Type III):

$$3r_2 + r_1 \qquad \begin{array}{rcrcrcr} & & 7x_2 & + & 11x_3 & = & 2 \\ -x_1 & + & 3x_2 & + & 6x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & 3 \end{array}$$

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As a consequence, performing a sequence of elementary operations on a system of linear equations results in an equivalent system of linear equations, with the exact same solutions.

The Augmented Matrix

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Represent a system of linear equations with its augmented matrix.

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The system of linear equations

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(A matrix is a rectangular array of numbers.)

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Remark

Two other matrices associated with a system of linear equations are the coefficient matrix and the constant matrix:

$$\begin{bmatrix} 1 & -2 & -7 \\ -1 & 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

For convenience, instead of performing elementary operations on a system of linear equations, perform corresponding elementary row operations on the corresponding augmented matrix.

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Type I: Interchange two rows.

Example

Interchange rows 1 and 3.

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{rrrr|r} 0 & 5 & -6 & 1 & 0 \\ -2 & 0 & 3 & 3 & -1 \\ 2 & -1 & 0 & 5 & -3 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right]$$

Type II: Multiply a row by a nonzero number.

Example

Multiply row 4 by 2.

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right] \xrightarrow{2r_4} \left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 2 & -8 & 4 & 4 & 4 \end{array}\right]$$

Type III: Add a multiple of one row to a different row.

Example

Add 2 times row 4 to row 2.

$$\left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ -2 & 0 & 3 & 3 & -1 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right] \xrightarrow{2r_4+r_2} \left[\begin{array}{rrrr|r} 2 & -1 & 0 & 5 & -3 \\ 0 & -8 & 7 & 7 & 3 \\ 0 & 5 & -6 & 1 & 0 \\ 1 & -4 & 2 & 2 & 2 \end{array}\right]$$

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Two matrices A and B are row equivalent (or simply equivalent) if one can be obtained from the other by a sequence of elementary row operations.

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Problem

Prove that A can be obtained from B by a sequence of elementary row operations if and only if B can be obtained from A by a sequence of elementary row operations.
Prove that row equivalence is an equivalence relation.

Solving a System using Back Substitution

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Solve the system using back substitution

$$\begin{aligned} 2x + y &= 4 \\ x - 3y &= 1 \end{aligned}$$

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The result is an equivalent system

$$\begin{aligned} 7y &= 2 \\ x - 3y &= 1 \end{aligned}$$

Solution (continued)

The first equation of the system,

$$7y = 2$$

can be rearranged to give us

$$y = \frac{2}{7}.$$

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$$x = 1 + \frac{6}{7} = \frac{13}{7}.$$

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We shall describe an algorithm for solving any given system of linear equations.