

Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations §1-3. Homogeneous Equations

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(last updated on 10/26/2020)



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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Homogeneous Equations

Linear Combination

Homogeneous Equations

Definition

A **homogeneous linear equation** is one whose constant term is equal to zero. A system of linear equations is called **homogeneous** if each equation in the system is homogeneous. A **homogeneous system** has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

where a_{ij} are scalars and x_i are variables, $1 \leq i \leq m$, $1 \leq j \leq n$.

Remark

1. Notice that $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution to a homogeneous system of equations. We call this the **trivial solution**.
2. We are interested in finding, if possible, **nontrivial solutions** (ones with at least one variable not equal to zero) to homogeneous systems.

Example

$$\text{Solve the system } \begin{cases} x_1 + x_2 - x_3 + 3x_4 = 0 \\ -x_1 + 4x_2 + 5x_3 - 2x_4 = 0 \\ x_1 + 6x_2 + 3x_3 + 4x_4 = 0 \end{cases}$$

Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 0 \\ -1 & 4 & 5 & -2 & 0 \\ 1 & 6 & 3 & 4 & 0 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -9/5 & 14/5 & 0 \\ 0 & 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

The system has infinitely many solutions, and the general solution is

$$\begin{cases} x_1 = \frac{9}{5}s - \frac{14}{5}t \\ x_2 = -\frac{4}{5}s - \frac{1}{5}t \\ x_3 = s \\ x_4 = t \end{cases} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{9}{5}s - \frac{14}{5}t \\ -\frac{4}{5}s - \frac{1}{5}t \\ s \\ t \end{bmatrix}, \forall s, t \in \mathbb{R}.$$

Theorem

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Linear Combination

Definition

If X_1, X_2, \dots, X_p are columns with the same number of entries, and if $a_1, a_2, \dots, a_p \in \mathbb{R}$ (are scalars) then $a_1X_1 + a_2X_2 + \dots + a_pX_p$ is a **linear combination** of columns X_1, X_2, \dots, X_p .

Example (continued)

In the previous example,

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} \frac{9}{5}s - \frac{14}{5}t \\ -\frac{4}{5}s - \frac{1}{5}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{9}{5}s \\ -\frac{4}{5}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{14}{5}t \\ -\frac{1}{5}t \\ 0 \\ t \end{bmatrix} \\ &= s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Example (continued)

This gives us

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} = sX_1 + tX_2,$$

$$\text{with } X_1 = \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix}.$$

The columns X_1 and X_2 are called **basic solutions** to the original homogeneous system.

Example (continued)

Notice that

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} = \frac{s}{5} \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix} + \frac{t}{5} \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix} \\ &= r \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix} + q \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix} \\ &= r(5X_1) + q(5X_2) \end{aligned}$$

where $r, q \in \mathbb{R}$.

Example (continued)

The columns $5X_1 = \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix}$ and $5X_2 = \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix}$ are also basic solutions to the original homogeneous system.

Remark

In general, any nonzero multiple of a basic solution (to a homogeneous system of linear equations) is also a basic solution.

What does the rank tell us in the homogeneous case?

Suppose A is the augmented matrix of an homogeneous system of m linear equations in n variables, and $\text{rank } A = r$.

$$m \left\{ \begin{array}{cccc|c} * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \\ * & * & * & * & 0 \end{array} \right\} \rightarrow \begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$\underbrace{\hspace{10em}}_n \qquad \qquad \qquad \underbrace{\hspace{10em}}_{r \text{ leading } 1\text{'s}}$

There is always a solution, and the set of solutions to the system has $n - r$ parameters, so

- ▶ if $r < n$, there is at least one parameter, and the system has infinitely many solutions;
- ▶ if $r = n$, there are no parameters, and the system has a unique solution, the trivial solution.

Theorem

Let A be an $m \times n$ matrix of rank r , and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. The system has exactly $n - r$ basic solutions, one for each parameter.
2. Every solution is a **linear combination** of these **basic solutions**.

Problem

Find all values of a for which the system

$$\begin{cases} x + y & = 0 \\ ay + z & = 0 \\ x + y + az & = 0 \end{cases}$$

has nontrivial solutions, and determine the solutions.

Solution

Non-trivial solutions occur only when $a = 0$, and the solutions when $a = 0$ are given by (rank $r = 2$, $n - r = 3 - 2 = 1$ parameter)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \forall s \in \mathbb{R}.$$