Math 221: LINEAR ALGEBRA

Chapter 1. Systems of Linear Equations §1-3. Homogeneous Equations

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Emory University, 2020 Fall

(last updated on 10/26/2020)



Linear Combination

Definition

A homogeneous linear equation is one whose constant term is equal to zero. A system of linear equations is called homogeneous if each equation in the system is homogeneous. A homogeneous system has the form

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Remark

- 1. Notice that $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is always a solution to a homogeneous system of equations. We call this the trivial solution.
- 2. We are interested in finding, if possible, nontrivial solutions (ones with at least one variable not equal to zero) to homogeneous systems.

Solve the system
$$\begin{cases} x_1 + x_2 - x_3 + 3x_4 = 0\\ -x_1 + 4x_2 + 5x_3 - 2x_4 = 0\\ x_1 + 6x_2 + 3x_3 + 4x_4 = 0 \end{cases}$$

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Solution

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 0 \\ -1 & 4 & 5 & -2 & 0 \\ 1 & 6 & 3 & 4 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cccc|c} 1 & 0 & -9/5 & 14/5 & 0 \\ 0 & 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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The system has infinitely many solutions, and the general solution is

$$\begin{cases} x_1 &= \frac{9}{5}s - \frac{14}{5}t \\ x_2 &= -\frac{4}{5}s - \frac{1}{5}t \\ x_3 &= s \\ x_4 &= t \end{cases}$$

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Theorem

If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution (in fact, infinitely many).

Definition

If X_1, X_2, \ldots, X_p are columns with the same number of entries, and if $a_1, a_2, \ldots a_p \in \mathbb{R}$ (are scalars) then $a_1X_1 + a_2X_2 + \cdots + a_pX_p$ is a linear combination of columns X_1, X_2, \ldots, X_p .

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Example (continued)

In the previous example,

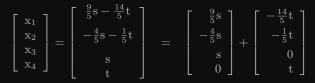
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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{9}{5}s - \frac{14}{5}t \\ -\frac{4}{5}s - \frac{1}{5}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} \frac{9}{5}s \\ -\frac{4}{5}s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{14}{5}t \\ -\frac{1}{5}t \\ 0 \\ t \end{bmatrix}$$
$$= s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix}$$

This gives us

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} = sX_1 + tX_2,$$

with $X_1 = \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix}.$

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The columns X_1 and X_2 are called **basic solutions** to the original homogeneous system.

Notice that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 9/5 \\ -4/5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -14/5 \\ -1/5 \\ 0 \\ 1 \end{bmatrix} = \frac{s}{5} \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix} + \frac{t}{5} \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix}$$
$$= r \begin{bmatrix} 9 \\ -4 \\ 5 \\ 0 \end{bmatrix} + q \begin{bmatrix} -14 \\ -1 \\ 0 \\ 5 \end{bmatrix}$$
$$= r(5X_1) + q(5X_2)$$

where $r, q \in \mathbb{R}$.

The columns
$$5X_1 = \begin{bmatrix} 9\\ -4\\ 5\\ 0 \end{bmatrix}$$
 and $5X_2 = \begin{bmatrix} -14\\ -1\\ 0\\ 5 \end{bmatrix}$ are also basic solutions

to the original homogeneous system.

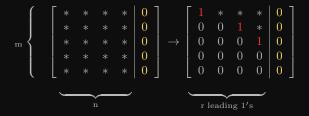
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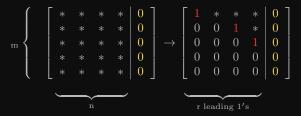
Remark

In general, any nonzero multiple of a basic solution (to a homogeneous system of linear equations) is also a basic solution.

Suppose A is the augmented matrix of an homogeneous system of m linear equations in n variables, and rank A = r.

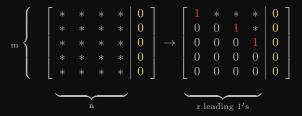


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There is always a solution, and the set of solutions to the system has $\mathbf{n}-\mathbf{r}$ parameters, so

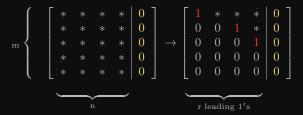
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- if r < n, there is at least one parameter, and the system has infinitely many solutions;
- if r = n, there are no parameters, and the system has a unique solution, the trivial solution.

Theorem

Let A be an $m\times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

- 1. The system has exactly n r basic solutions, one for each parameter.
- 2. Every solution is a linear combination of these basic solutions.

Problem

Find all values of a for which the system

$$\begin{cases} x + y &= 0\\ & ay + z &= 0\\ x + y + az &= 0 \end{cases}$$

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Solution

Non-trivial solutions occur only when a = 0, and the solutions when a = 0 are given by (rank r = 2, n - r = 3 - 2 = 1 parameter)

$$\left[\begin{array}{c} x\\ y\\ z\end{array}\right]=s\left[\begin{array}{c} 1\\ -1\\ 0\end{array}\right],\quad \forall s\in\mathbb{R}.$$