

Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra

§2-2. Equations, Matrices, and Transformations

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Vectors

Matrix Vector Multiplication

The Dot Product

Transformations

Rotations in \mathbb{R}^2

Vectors

Definitions

A row matrix or column matrix is often called a **vector**, and such matrices are referred to as **row vectors** and **column vectors**, respectively. If \vec{x} is a **row vector** of size $1 \times n$, and \vec{y} is a **column vector** of size $m \times 1$, then we write

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Definition (Vector form of a system of linear equations)

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Such a system can be expressed in **vector form** or as a **vector equation** by using **linear combinations** of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Problem

Express the following system of linear equations in vector form:

$$\begin{array}{cccccc} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

Solution

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Matrix vector multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$, written $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$, and let \vec{x} be an $n \times 1$ column vector,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then **the product of matrix A and (column) vector \vec{x}** is the $m \times 1$ column vector given by

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \sum_{j=1}^n x_j \vec{a}_j$$

that is, $A\vec{x}$ is a **linear combination** of the columns of A.

Problem

Compute the product $A\vec{x}$ for

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solution

$$A\vec{x} = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

Problem

Compute $A\vec{y}$ for

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

Solution

$$A\vec{y} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

Definition (Matrix form of a system of linear equations)

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Such a system can be expressed in **matrix form** using matrix vector multiplication,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus a system of linear equations can be expressed as a **matrix equation**

$$A\vec{x} = \vec{b},$$

where A is the coefficient matrix, \vec{b} is the constant matrix, and \vec{x} is the matrix of variables.

Problem

Express the following system of linear equations in matrix form.

$$\begin{array}{rrcrcl} 2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\ & & - & x_2 & + & 5x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array}$$

Solution

$$\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Theorem

1. Every system of m linear equations in n variables can be written in the form $A\vec{x} = \vec{b}$ where A is the coefficient matrix, \vec{x} is the matrix of variables, and \vec{b} is the constant matrix.

Theorem (continued)

2. The system $A\vec{x} = \vec{b}$ is consistent (i.e., has at least one solution) if and only if \vec{b} is a linear combination of the columns of A .

Theorem (continued)

3. The vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the system $A\vec{x} = \vec{b}$ if and only

if x_1, x_2, \dots, x_n are a solution to the vector equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots x_n \vec{a}_n = \vec{b}$$

where $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ are the columns of A .

Problem

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express \vec{b} as a linear combination of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ of A , or show that this is impossible.

Solution

Solve the system $A\vec{x} = \vec{b}$ where \vec{x} is a column vector with four entries. Do so by putting the augmented matrix $\left[A \mid \vec{b} \right]$ in reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1/7 \\ 0 & 1 & 0 & 1 & -5/7 \\ 0 & 0 & 1 & -1 & 3/7 \end{array} \right]$$

Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4 . Choosing $x_4 = 0$ (which is the simplest thing to do) gives us

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{7} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{7} \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{7}\vec{a}_1 - \frac{5}{7}\vec{a}_2 + \frac{3}{7}\vec{a}_3 + 0\vec{a}_4.$$



Remark

The problem may ask to find **all possible** linear combinations of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ of A.

This is equivalent to find all solutions to the corresponding system of linear equations:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} - s \\ -\frac{5}{7} - s \\ \frac{3}{7} + s \\ s \end{bmatrix}$$

Hence, all possible linear combinations are:

$$\vec{b} = \left(\frac{1}{7} - s\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{5}{7} + s\right) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{7} + s\right) \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Theorem

Let A and B be $m \times n$ matrices, and let \vec{x} and \vec{y} be n -vectors in \mathbb{R}^n . Then:

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.
2. $A(a\vec{x}) = a(A\vec{x}) = (aA)\vec{x}$ for all scalars a .
3. $(A + B)\vec{x} = A\vec{x} + B\vec{x}$.

This provides a useful way to describe the solutions to a system $A\vec{x} = \vec{b}$.

Theorem

Suppose \vec{x}_1 is any particular solution to the system $A\vec{x} = \vec{b}$ of linear equations. Then every solution \vec{x}_2 to $A\vec{x} = \vec{b}$ has the form $\vec{x}_2 = \vec{x}_0 + \vec{x}_1$ for some solution \vec{x}_0 of the associated homogeneous system $A\vec{x} = \vec{0}$.

The Dot Product

Definition

If (a_1, a_2, \dots, a_n) and (b_1, b_2, \dots, b_n) are two ordered n -tuples, their **dot product** is defined to be the number

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

obtained by multiplying corresponding entries and adding the results.

This is very much related to the matrix product $A\vec{x}$.

Theorem (Dot Product Rule)

Let A be an $m \times n$ matrix and let \vec{x} be an n -vector. Then each entry of the vector $A\vec{x}$ is the dot product of the corresponding row of A with \vec{x} .


Problem

$$\text{If } A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \text{ compute } A\vec{x}.$$

Solution

The entries of $A\vec{x}$ are the dot products of the rows of A with \vec{x} :

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 & + & 0(-1) & + & 2 \cdot 1 & + & (-1)4 \\ 2 \cdot 2 & + & (-1)(-1) & + & 0 \cdot 1 & + & 1 \cdot 4 \\ 3 \cdot 2 & + & 1(-1) & + & 3 \cdot 1 & + & 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}. \end{aligned}$$

Of course, this agrees with the outcome of the previous example. 

Definition (Identity Matrix)

For each $n > 2$, the **identity matrix** I_n is the $n \times n$ matrix with 1's on the main diagonal (upper left to lower right), and zeros elsewhere.

Example

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \dots$$


Problem

Show that $I_n \vec{x} = \vec{x}$ for each n -vector \vec{x} in \mathbb{R}^n , $n \geq 1$.

Solution

We verify the case $n = 4$. Given the 4-vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4 \vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + 0 + 0 \\ 0 + x_2 + 0 + 0 \\ 0 + 0 + x_3 + 0 \\ 0 + 0 + 0 + x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x}.$$

In general, $I_n \vec{x} = \vec{x}$ because entry k of $I_n \vec{x}$ is the dot product of row k of I_n with \vec{x} , and row k of I_n has 1 in position k and zeros elsewhere. 

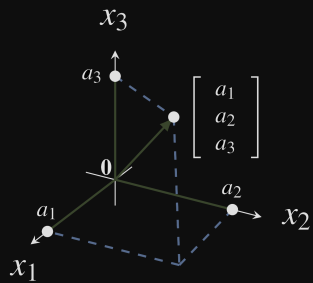
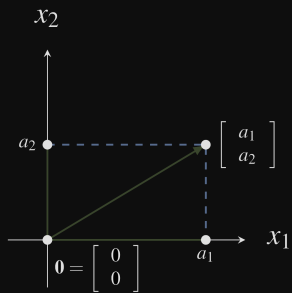
Transformations

Notation and Terminology

- ▶ We have already used \mathbb{R} to denote the set of real numbers.
- ▶ We use \mathbb{R}^2 to denote the set of all column vectors of length two, and we use \mathbb{R}^3 to denote the set of all column vectors of length three (the length of a vector is the number of entries it contains).
- ▶ In general, we write \mathbb{R}^n for the set of all column vectors of length n .

\mathbb{R}^2 and \mathbb{R}^3

Vectors in \mathbb{R}^2 and \mathbb{R}^3 have convenient geometric interpretations as position vectors of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.



Definition (Transformations)

A **transformation** is a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, sometimes written $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, and is called a **transformation from \mathbb{R}^n to \mathbb{R}^m** . If $m = n$, then we say **T is a transformation of \mathbb{R}^n** .

What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a rule that, for each vector in \mathbb{R}^n , assigns exactly one vector of \mathbb{R}^m .

We use the notation $T(\vec{x})$ to mean the transformation T applied to the vector \vec{x} .

Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a **matrix transformation**, and write $T_A(\vec{x}) = A\vec{x}$.

Definition (Equality of Transformations)

Suppose $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Example (Specifying the action of a transformation)

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}$$

is a transformation that **transforms** the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in \mathbb{R}^3 into the vector

$$T \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 + 4 \\ 4 + 7 \\ 1 - 7 \\ 7 - 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

Example (Transformation by matrix multiplication)

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A

transforms vectors in \mathbb{R}^3 into vectors in \mathbb{R}^2 . Consider the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Transforming this vector by A looks like:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}.$$

For example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Rotations in \mathbb{R}^2

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the **matrix transformation induced by A** .

Definition

The transformation

$$R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

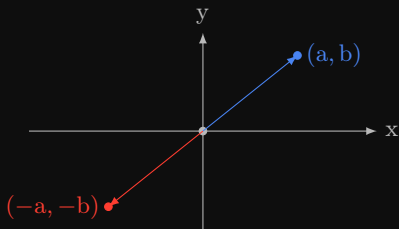
denotes **counterclockwise rotation** about the origin through an angle of θ .

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π .



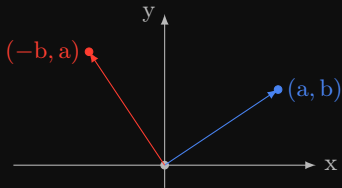
We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Remark

In general, the rotation (counterclockwise) about the origin for an angle θ is

$$R_{\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \cos(\theta) - b \sin(\theta) \\ a \sin(\theta) + b \cos(\theta) \end{bmatrix}$$

$$R_{\pi} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad R_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$