Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra

§2-2. Equations, Matrices, and Transformations

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Emory University, 2020 Fall

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Vectors

Vectors

Definitions

A row matrix or column matrix is often called a vector, and such matrices are referred to as row vectors and column vectors, respectively. If $\vec{x}$ is a row vector of size $1 \times n$, and $\vec{y}$ is a column vector of size $m \times 1$, then we write

$$\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Definition (Vector form of a system of linear equations)

Consider the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

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Such a system can be expressed in vector form or as a vector equation by using linear combinations of column vectors:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Problem

Express the following system of linear equations in vector form:

$$
\begin{array}{rcrcrcr}
2x_1 & + & 4x_2 & - & 3x_3 & = & -6 \\
 & - & x_2 & + & 5x_3 & = & 0 \\
x_1 & + & x_2 & + & 4x_3 & = & 1
\end{array}
$$

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Solution

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Matrix vector multiplication

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Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix with columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, written $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$, and let $\vec{x}$ be an $n \times 1$ column vector,

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$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product of matrix A and (column) vector $\vec{x}$ is the $m \times 1$ column vector given by

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \sum_{j=1}^{n} x_j\vec{a}_j$$

that is, $A\vec{x}$ is a linear combination of the columns of A.

Problem

Compute the product $A\vec{x}$ for

$$A = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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Solution

$$A\vec{x} = \begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

Problem

Compute $A\vec{y}$ for

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

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Solution

$$A\vec{y} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

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Such a system can be expressed in matrix form using matrix vector multiplication,

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Thus a system of linear equations can be expressed as a matrix equation

$$A\vec{x} = \vec{b},$$

where A is the coefficient matrix, $\vec{b}$ is the constant matrix, and $\vec{x}$ is the matrix of variables.

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Solution

$$\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 5 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

Theorem

1. Every system of m linear equations in n variables can be written in the form $A\vec{x} = \vec{b}$ where A is the coefficient matrix, $\vec{x}$ is the matrix of variables, and $\vec{b}$ is the constant matrix.

Theorem (continued)

2. The system $A\vec{x} = \vec{b}$ is consistent (i.e., has at least one solution) if and only if $\vec{b}$ is a linear combination of the columns of A.

Theorem (continued)

3. The vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the system $A\vec{x} = \vec{b}$ if and only if $x_1, x_2, \ldots, x_n$ are a solution to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n = \vec{b}$$

where $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the columns of A.

Problem

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Express $\vec{b}$ as a linear combination of the columns $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ of A, or show that this is impossible.

Solution

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$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 1/7 \\ 0 & 1 & 0 & 1 & -5/7 \\ 0 & 0 & 1 & -1 & 3/7 \end{array}\right]$$

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Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4.

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Since there are infinitely many solutions (x_4 is assigned a parameter), choose any value for x_4. Choosing $x_4 = 0$ (which is the simplest thing to do) gives us

$$\vec{b} = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] = \frac{1}{7}\left[\begin{array}{r} 1 \\ 2 \\ 3 \end{array}\right] - \frac{5}{7}\left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right] + \frac{3}{7}\left[\begin{array}{r} 2 \\ 0 \\ 3 \end{array}\right] = \frac{1}{7}\vec{a}_1 - \frac{5}{7}\vec{a}_2 + \frac{3}{7}\vec{a}_3 + 0\vec{a}_4.$$

$\blacksquare$

Remark

The problem may ask to to find all possible linear combinations of the columns $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_4$ of A.

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This is equivalent to find all solutions to the corresponding system of linear equations:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} - s \\ -\frac{5}{7} - s \\ \frac{3}{7} + s \\ s \end{bmatrix}$$

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Hence, all possible linear combinations are:

$$\vec{b} = \left(\frac{1}{7} - s\right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{5}{7} + s\right) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{3}{7} + s\right) \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Theorem

Let A and B be $m \times n$ matrices, and let $\vec{x}$ and $\vec{y}$ be n-vectors in $\mathbb{R}^n$. Then:

1. $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$.
2. $A(a\vec{x}) = a(A\vec{x}) = (aA)\vec{x}$ for all scalars a.
3. $(A + B)\vec{x} = A\vec{x} + B\vec{x}$.

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Theorem

Suppose $\vec{x_1}$ is any particular solution to the system $A\vec{x} = \vec{b}$ of linear equations. Then every solution $\vec{x_2}$ to $A\vec{x} = \vec{b}$ has the form $\vec{x_2} = \vec{x_0} + \vec{x_1}$ for some solution $\vec{x_0}$ of the associated homogeneous system $A\vec{x} = \vec{0}$.

The Dot Product

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Definition

If $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are two ordered n-tuples, their dot product is defined to be the number

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

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This is very much related ot the matrix produxt $A\vec{x}$.

Theorem (Dot Product Rule)

Let A be an $m \times n$ matrix and let $\vec{x}$ be an n-vector. Then each entry of the vector $A\vec{x}$ is the dot product of the corresponding row of A with $\vec{x}$.

Problem

If $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$, compute $A\vec{x}$.

Problem

If $\mathrm{A} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$ and $\vec{\mathrm{x}} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$, compute $\mathrm{A}\vec{\mathrm{x}}$.

Solution

The entries of $\mathrm{A}\vec{\mathrm{x}}$ are the dot products of the rows of A with $\vec{\mathrm{x}}$:

$$\begin{aligned} \mathrm{A}\vec{\mathrm{x}} &= \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 2 & + & 0(-1) & + & 2 \cdot 1 & + & (-1)4 \\ 2 \cdot 2 & + & (-1)(-1) & + & 0 \cdot 1 & + & 1 \cdot 4 \\ 3 \cdot 2 & + & 1(-1) & + & 3 \cdot 1 & + & 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}. \end{aligned}$$

Of course, this agrees with the outcome of the previous example. ∎

Definition (Identity Matrix)

For each $n > 2$, the identity matrix I_n is the $n \times n$ matrix with 1's on the main diagonal (upper left to lower right), and zeros elsewhere.

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Example

The first few identity matrices are

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \ldots$$

Problem

Show that $I_n\vec{x} = \vec{x}$ for each n-vector $\vec{x}$ in $\mathbb{R}^n$, $n \geq 1$.

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Solution

We verify the case $n = 4$. Given the 4-vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ the dot product rule gives

$$I_4\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + 0 + 0 \\ 0 + x_2 + 0 + 0 \\ 0 + 0 + x_3 + 0 \\ 0 + 0 + 0 + x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{x}.$$

In general, $I_n\vec{x} = \vec{x}$ because entry k of $I_n\vec{x}$ is the dot product of row k of I_n with $\vec{x}$, and row k of I_n has 1 in position k and zeros elsewhere. ■

Transformations

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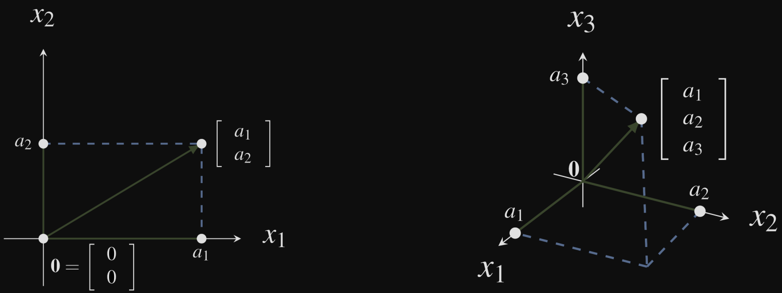
$\mathbb{R}^2$ and $\mathbb{R}^3$

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have convenient geometric interpretations as position vectors of points in the 2-dimensional (Cartesian) plane and in 3-dimensional space, respectively.


x_2
a_2
$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$
x_1
a_1
$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
x_3
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Definition (Transformations)

A **transformation** is a function $T : \mathbb{R}^n \to \mathbb{R}^m$, sometimes written $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$, and is called a **transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$**.

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What do we mean by a function?

Informally, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that, for each vector in $\mathbb{R}^n$, assigns exactly one vector of $\mathbb{R}^m$

We use the notation $T(\vec{x})$ to mean the transformation T applied to the vector $\vec{x}$.

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Definition

If T acts by matrix multiplication of a matrix A (such as the previous example), we call T a matrix transformation, and write $T_A(\vec{x}) = A\vec{x}$.

Definition (Equality of Transformations)

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\vec{x}) = T(\vec{x})$ for every $\vec{x} \in \mathbb{R}^n$.

Example (Specifying the action of a transformation)

$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}$$

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is a transformation that transforms the vector $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ in $\mathbb{R}^3$ into the vector

$$T\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 4+7 \\ 1-7 \\ 7-4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ -6 \\ 3 \end{bmatrix}.$$

Example (Transformation by matrix multiplication)

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. By matrix multiplication, A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

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Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

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Definition

The transformation

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

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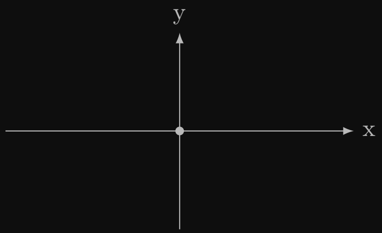

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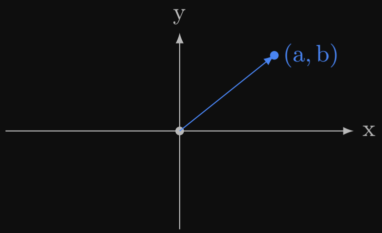

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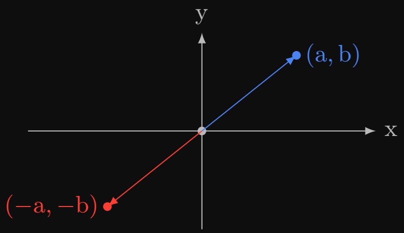

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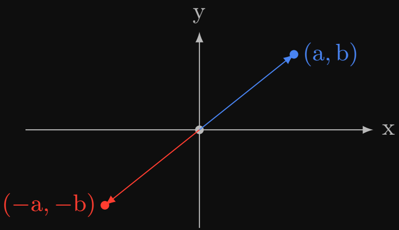

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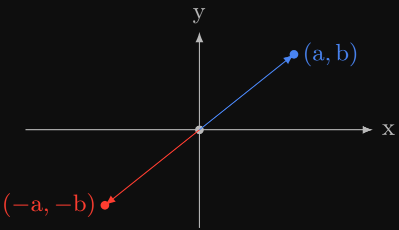

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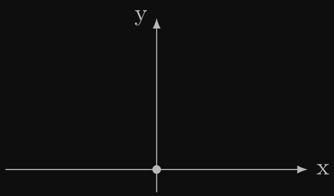

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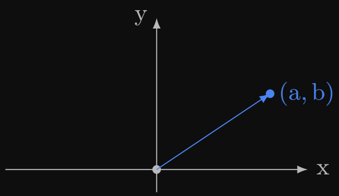

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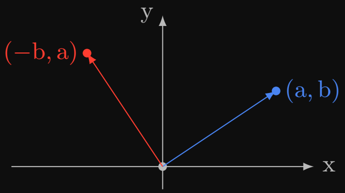

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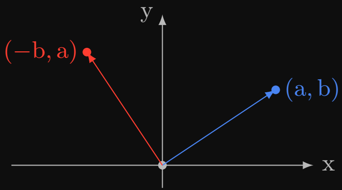

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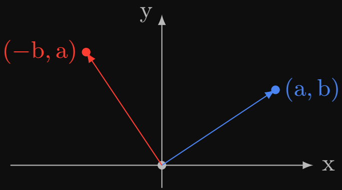

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Remark

In general, the rotation (counterclockwise) about the origin for an angle θ is

$$\mathrm{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

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