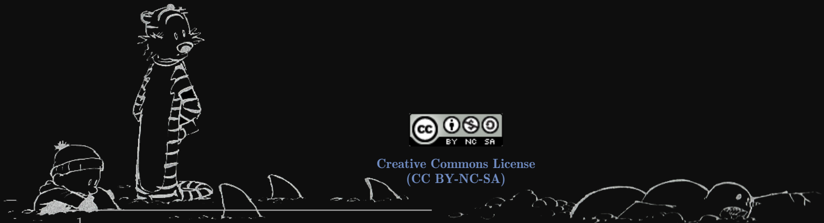
Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-3. Matrix Multiplication

Le Chen[1]

Emory University, 2020 Fall

(last updated on 10/26/2020)



1 Slides are adapted from those by Karen Seyffarth from University of Calgary.

Matrix Multiplication

Properties of Matrix Multiplication

Matrix Multiplication

Definition (Product of two matrices)

Let A be an $m \times n$ matrix and let $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix}$ be an $n \times p$ matrix, whose columns are $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p$. The product of A and B is the matrix

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}$$

i.e., the first column of AB is $A\vec{b}_1$, the second column of AB is $A\vec{b}_2$, etc. Note that AB has size $m \times p$.

Problem

Find the product AB of matrices

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution

AB has columns

$$A\vec{b}_1 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix},$$

$$A\vec{b}_2 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

$$A\vec{b}_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

Thus, $AB = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}$.

Definition

Let A and B be matrices, and suppose that A is $m \times n$.

- In order for the product AB to exist, the number of rows in B must be equal to the number of columns in A, implying that B is an $n \times p$ matrix for some p.
- When defined, AB is an $m \times p$ matrix.

If the product is defined, then A and B are said to be compatible for (matrix) multiplication.

Example

As we saw in the previous problem

$$\overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}} \overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{2\times 3}{\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}}$$

Note that the product

$$\overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} \overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}}$$

does not exist.

Example (Multiplication by the zero matrix)

Compute the product AO for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad AO = O.$$

∎

Definition (The (i, j)-entry of a product)

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the (i, j)-entry of AB is given by the dot product of row i of A and column j of B:

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$\overset{m \times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \overset{n \times s}{\begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2s} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{ns} \end{bmatrix}} = \overset{m \times s}{\begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1s} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{is} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{ms} \end{bmatrix}}$$

Example

Using the above definition, the $(2, 3)$-entry of the product

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

is computed by the dot product of the second row of the first matrix and the third column of the second matrix:

$$2 \times 2 + (-1) \times 4 + 1 \times 0 = 4 - 4 + 0 = 0.$$

Properties of Matrix Multiplication

Given matrices A and B, is $AB = BA$?

Problem

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

Solution

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

BA does not exist!

Problem

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

Solution

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$

Remark

In this example, GH and HG both exist, but they are not equal. They aren't even the same size!

Problem

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

Solution

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$

■

Remark

In this example, PQ and QP both exist and are the same size, but $PQ \neq QP$.

Fact

The three preceding problems illustrate an important property of matrix multiplication.

> In general, matrix multiplication is **not** commutative, i.e., the order of the matrices in the product is important.

In other words, in general

$$AB \neq BA.$$

Multiplying from left or right, it MATTERS!

Problem

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

Solution

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

■

Remark

In this particular example, the matrices commute, i.e., $UV = VU$.

Theorem (Properties of Matrix Multiplication)

Let A, B, and C be matrices of the appropriate sizes, and let $r \in \mathbb{R}$ be a scalar. Then the following properties hold.

1. $IA = A$ and $AI = A$.
2. $A(B + C) = AB + AC$.
   (matrix multiplication distributes over matrix addition).
3. $(B + C)A = BA + CA$.
   (matrix multiplication distributes over matrix addition).
4. $A\,(BC) = (AB)\,C$. (matrix multiplication is associative).
5. $r(AB) = (rA)B = A(rB)$.
6. $(AB)^T = B^T A^T$.

Remark

This applies to matrix-vector multiplication as well, since a vector is a row matrix or a column matrix.

Problem

Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be three $n \times n$ matrices. For $1 \leq i, j \leq n$ write down a formula for the (i, j)-entry of each of the following matrices.

1. AB
2. BA
3. $A+C$
4. $C(A+B)$
5. $A(BC)$
6. $(AB)C$

Problem

Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix. Prove that if A and B commute with C, then $A + B$ commutes with C.

Proof.

We are given that $AC = CA$ and $BC = CB$. Consider $(A + B)C$.

$$\begin{aligned}(A+B)C &= AC + BC \\ &= CA + CB \\ &= C(A+B)\end{aligned}$$

Since $(A + B)C = C(A + B)$, $A + B$ commutes with C. ■

Problem

Let A, B and C be $n \times n$ matrices, and suppose that both A and B commute with C, i.e., $AC = CA$ and $BC = CB$. Show that AB commutes with C.

Proof.

We must show that $(AB)C = C(AB)$ given that $AC = CA$ and $BC = CB$.

$$
\begin{aligned}
(AB)C &= A(BC) && \text{(matrix multiplication is associative)} \\
&= A(CB) && \text{(B commutes with C)} \\
&= (AC)B && \text{(matrix multiplication is associative)} \\
&= (CA)B && \text{(A commutes with C)} \\
&= C(AB) && \text{(matrix multiplication is associative)}
\end{aligned}
$$

Therefore, AB commutes with C. ■

Partitioned matrix and block multiplication

Observation

We can partition matrix into blocks so that each entry of the partitioned matrix is again a matrix.

Example

$$A = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 2 & -1 & 4 & 2 & 1 \\ 3 & 1 & -1 & 7 & 5 \end{array}\right] = \begin{bmatrix} I_2 & 0_{23} \\ P & Q \end{bmatrix} \quad \text{and} \quad B = \left[\begin{array}{cc} 4 & -2 \\ 5 & 6 \\ \hline 7 & 3 \\ -1 & 0 \\ 1 & 6 \end{array}\right] = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Example

Let A and B be $m \times n$ and $n \times k$ matrices, respectively. We can partition then into either column vectors or row vectors: When viewed as partitioned matrices, AB can be equivalently written in one of the following four ways:

$$A_{mn} = (\vec{a}_1, \cdots, \vec{a}_n) = \begin{pmatrix} \vec{\alpha}_1^T \\ \vdots \\ \vec{\alpha}_m^T \end{pmatrix} \quad \text{and} \quad B_{nk} = \left(\vec{b}_1, \cdots, \vec{b}_k\right) = \begin{pmatrix} \vec{\beta}_1^T \\ \vdots \\ \vec{\beta}_n^T \end{pmatrix}$$

1. $$AB = A\left(\vec{b}_1, \cdots, \vec{b}_k\right) = \left(A\vec{b}_1, \cdots, A\vec{b}_k\right)$$

2. $$AB = \begin{pmatrix} \vec{\alpha}_1^T \\ \vdots \\ \vec{\alpha}_m^T \end{pmatrix} B = \begin{pmatrix} \vec{\alpha}_1^T B \\ \vdots \\ \vec{\alpha}_m^T B \end{pmatrix}$$

Example (continued)

3

$$\mathrm{AB} = (\vec{\mathrm{a}}_1, \cdots, \vec{\mathrm{a}}_\mathrm{n}) \begin{pmatrix} \vec{\beta}_1^\mathrm{T} \\ \vdots \\ \vec{\beta}_\mathrm{n}^\mathrm{T} \end{pmatrix} = \vec{\mathrm{a}}_1 \vec{\beta}_1^\mathrm{T} + \vec{\mathrm{a}}_2 \vec{\beta}_2^\mathrm{T} + \cdots \vec{\mathrm{a}}_\mathrm{n} \vec{\beta}_\mathrm{n}^\mathrm{T}$$

4

$$\mathrm{AB} = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T} \\ \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T} \end{pmatrix} \left(\vec{\mathrm{b}}_1, \cdots, \vec{\mathrm{b}}_\mathrm{k}\right) = \begin{pmatrix} \vec{\alpha}_1^\mathrm{T}\mathrm{b}_1 & \vec{\alpha}_1^\mathrm{T}\mathrm{b}_2 & \cdots & \vec{\alpha}_1^\mathrm{T}\mathrm{b}_\mathrm{k} \\ \vec{\alpha}_2^\mathrm{T}\mathrm{b}_1 & \vec{\alpha}_2^\mathrm{T}\mathrm{b}_2 & \cdots & \vec{\alpha}_2^\mathrm{T}\mathrm{b}_\mathrm{k} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\alpha}_\mathrm{m}^\mathrm{T}\mathrm{b}_1 & \vec{\alpha}_\mathrm{m}^\mathrm{T}\mathrm{b}_\mathrm{m} & \cdots & \vec{\alpha}_\mathrm{m}^\mathrm{T}\mathrm{b}_\mathrm{k} \end{pmatrix}$$

Example (continued)

One can also partition A and B as follows:

$$\mathrm{A} = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} \\ \mathrm{A}_{21} & \mathrm{A}_{22} \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} \mathrm{B}_{11} & \mathrm{B}_{12} \\ \mathrm{B}_{21} & \mathrm{B}_{22} \end{pmatrix}$$

in a way that dimensions match. Then

$$\mathrm{AB} = \begin{pmatrix} \mathrm{A}_{11}\mathrm{B}_{11} + \mathrm{A}_{12}\mathrm{B}_{21} & \mathrm{A}_{11}\mathrm{B}_{12} + \mathrm{A}_{12}\mathrm{B}_{22} \\ \mathrm{A}_{21}\mathrm{B}_{11} + \mathrm{A}_{22}\mathrm{B}_{21} & \mathrm{A}_{21}\mathrm{B}_{12} + \mathrm{A}_{22}\mathrm{B}_{22} \end{pmatrix}$$

Problem

Let A be a square matrix. Compute A^k where $A = \begin{pmatrix} I & X \\ O & O \end{pmatrix}$.

Solution

$$A^2 = \cdots = A.$$

Hence, $A^k = A$ for all $k \geq 2$.