Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-4. Matrix Inverses

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

The Identity and Inverse Matrices

Finding the Inverse of a Matrix

Properties of the Inverse

Inverse of Transformations

The Identity and Inverse Matrices

Definition

For each $n \geq 2$, the $n \times n$ identity matrix, denoted I_n, is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

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Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition

Let $n \geq 2$. For each j, $1 \leq j \leq n$, we denote by $\vec{e}_j$ the j^{th} column of I_n.

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Example

When $n = 3$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Theorem

Let A be an $m \times n$ matrix. Then $AI_n = A$ and $I_mA = A$.

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Proof.

The (i, j)-entry of AI_n is the product of the i^{th} row of $A = [a_{ij}]$, namely $\left[\begin{array}{cccccc} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{array}\right]$ with the j^{th} column of I_n, namely $\vec{e}_j$. Since $\vec{e}_j$ has a one in row j and zeros elsewhere,

$$\left[\begin{array}{cccccc} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \end{array}\right] \vec{e}_j = a_{ij}$$

Since this is true for all $i \leq m$ and all $j \leq n$, $AI_n = A$.

The proof of $I_m A = A$ is analogous—work it out! ■

Instead of AI_n and I_mA we often write AI and IA, respectively, since the size of the identity matrix is clear from the context: the sizes of A and I must be compatible for matrix multiplication.

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Thus

$$AI = A \quad \text{and} \quad IA = A$$

which is why I is called an identity matrix – it is an identity for matrix multiplication.

Definition (Matrix Inverses)

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Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, B is an inverse of A.

Problem

Does every square matrix have an inverse?

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Solution

No! Take e.g. the zero matrix $\mathbf{O_n}$ (all entries of $\mathbf{O_n}$ are equal to 0)

$$\mathrm{A}\mathbf{O_n} = \mathbf{O_n}\mathrm{A} = \mathbf{O_n}$$

for all $\mathrm{n} \times \mathrm{n}$ matrices A:

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$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$$

which is never equal to I_2.

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If A is a square matrix and B and C are inverses of A, then B = C.

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Proof.

Since B and C are inverses of A, $AB = I = BA$ and $AC = I = CA$. Then

$$C = CI = C(AB) = CAB$$

and

$$B = IB = (CA)B = CAB$$

so $B = C$. ∎

Example (revisited)

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, we saw that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The preceding theorem tells us that B is the inverse of A, rather than just an inverse of A.

Remark (notation)

Let A be a square matrix, i.e., an $n \times n$ matrix.

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- If A has an inverse, then we say that A is invertible.

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$$\begin{aligned} AA^{-1} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

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Showing that $A^{-1}A = I_2$ is left as an exercise.

Remark

Here are some terminology related to this example:

1. Determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - cd$$

2. Adjugate:

$$\operatorname{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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Although the formula for the inverse of a 2×2 matrix is quicker and easier to use than the matrix inversion algorithm, the general formula for the inverse an $n \times n$ matrix, $n \geq 3$ (which we will see later), is more complicated and difficult to use than the matrix inversion algorithm. To find inverses of square matrices that are not 2×2, the matrix inversion algorithm is the most efficient method to use.

The Matrix Inversion Algorithm

Let A be an $n \times n$ matrix. To find A^{-1}, if it exists,

Step 1 take the $n \times 2n$ matrix

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$

obtained by augmenting A with the $n \times n$ identity matrix, I_n.

Step 2 Perform elementary row operations to transform $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ into a reduced row-echelon matrix.

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Theorem (Matrix Inverses)

Let A be an $n \times n$ matrix. Then the following conditions are equivalent.

1. A is invertible.
2. the reduced row-echelon form on A is I.
3. $\left[\begin{array}{c|c} A & I_n \end{array}\right]$ can be transformed into $\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$ using the Matrix Inversion Algorithm.

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Find, if possible, the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$.

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$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

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$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

From this, we see that A has no inverse.

Problem

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.

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$$[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] = [\,I \mid A^{-1}\,]$$

Solution (continued)

Therefore, A^{-1} exists, and

$$A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}$$

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You can check your work by computing AA^{-1} and $A^{-1}A$.

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Example

The system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

can be written in matrix form as $A\vec{x} = \vec{b}$:

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You can check that $A^{-1} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$.

Example (continued)

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i.e., $A\vec{x} = \vec{b}$ has the unique solution given by $\vec{x} = A^{-1}\vec{b}$. Therefore,

$$\vec{x} = A^{-1}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

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You should verify that $x = -2$, $y = -1$ is a solution to the system. ∎

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Problem

Can you find square matrices A, B and C for which $AB = AC$ but $B \neq C$?

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$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T$$

Properties of the Inverse

Example

Suppose A is an invertible matrix. Then

$$\mathrm{A}^{\mathrm{T}}(\mathrm{A}^{-1})^{\mathrm{T}} = (\mathrm{A}^{-1}\mathrm{A})^{\mathrm{T}} = \mathrm{I}^{\mathrm{T}} = \mathrm{I}$$

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3. If $A_1, A_2, \ldots, A_k$ are invertible, then $A_1A_2 \cdots A_k$ is invertible and
$$(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$$

(the third part is proved by iterating the above, or, more formally, by using the mathematical induction)

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4. If A is invertible and $p \in \mathbb{R}$ is nonzero, then pA is invertible, and $(pA)^{-1} = \frac{1}{p} A^{-1}$.

Example

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-\mathrm{A}^{\mathrm{T}} &= \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ -1 & -\frac{5}{2} \end{bmatrix} \\
\mathrm{A} &= \begin{bmatrix} \frac{3}{2} & 1 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix}
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True or false? Justify your answer.

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Therefore, A is invertible, and $A^{-1} = \frac{1}{4}A^2$. ■

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(6) $\Rightarrow$ (7) is clear. As for (7) $\Rightarrow$ (8), let $\vec{c}_j$ be one of the solution of $A\vec{x} = \vec{e}_j$. The

$$A[\vec{c}_1, \cdots, \vec{c}_n] = [\vec{e}_1, \cdots, \vec{e}_n] = I$$

Hence, (8) holds with $C = [\vec{c}_1, \cdots, \vec{c}_n]$.

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(9) $\Rightarrow$ (1): By reversing the roles of A and C in the previous argument, (9) implies that $C\vec{x} = \vec{0}$ has only the trivial solution, and we already know that this implies C is invertible. Thus A is the inverse of C, and hence A is itself invertible. ∎

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Corollary

If A and B are $n \times n$ matrices such that $AB = I$, then $BA = I$. Furthermore, A and B are invertible, with $B = A^{-1}$ and $A = B^{-1}$.

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Remark

Important Fact In Corollary, it is essential that the matrices be square.

Theorem

If A and B are matrices such that $AB = I$ and $BA = I$, then A and B are square matrices (of the same size).

Example

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

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$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \neq I_3.$$

Remark

This example illustrates why "an inverse" of a non-square matrix doesn't make sense. If A is $m \times n$ and B is $n \times m$, where $m \neq n$, then even if $AB = I$, it will never be the case that $BA = I$.

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Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix transformation induced by matrix A. Then A is invertible if and only if T has an inverse. In the case where T has an inverse, the inverse is unique and is denoted T^{-1}. Furthermore, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is induced by the matrix A^{-1}.

Fundamental Identities relating T and T^{-1}

1. $T^{-1} \circ T = 1_{\mathbb{R}^n}$
2. $T \circ T^{-1} = 1_{\mathbb{R}^n}$

Example

Let $\mathrm{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a transformation given by

$$\mathrm{T}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

Then T is a linear transformation induced by $\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

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Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a transformation given by

$$T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} x + y \\ y \end{array}\right]$$

Then T is a linear transformation induced by $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$.

Notice that the matrix A is invertible. Therefore the transformation T has an inverse, T^{-1}, induced by

$$A^{-1} = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array}\right]$$

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Therefore,

$$T^{-1}\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

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