Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-5. Elementary Matrices

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Elementary Matrices

Inverses of elementary matrices

Smith Normal Form

Elementary Matrices

Definition

An elementary matrix is a matrix obtained from an identity matrix by performing a single elementary row operation. The type of an elementary matrix is given by the type of row operation used to obtain the elementary matrix.

Remark

Three Types of Elementary Row Operations

- Type I: Interchange two rows.
- Type II: Multiply a row by a nonzero number.
- Type III: Add a (nonzero) multiple of one row to a different row.

Example

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

are examples of elementary matrices of types I, II and III, respectively. Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$$

We are interested in the effect that (left) multiplication of A by E, F and G has on the matrix A. Computing EA, FA, and GA ...

Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix}$$

$$FA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ -6 & -6 \\ 4 & 4 \end{bmatrix}$$

$$GA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$$

Remark

Notice that EA is the matrix obtained from A by interchanging row 2 and row 4, which is the same row operation used to obtain E from I_4. What about FA and GA?

Theorem (Multiplication by an Elementary Matrix)

Let A be an $m \times n$ matrix.

If B is obtained from A by performing one single elementary row operation,

then $B = EA$

where E is the elementary matrix obtained from I_m by performing the same elementary operation on I_m as was performed on A.

$$\begin{array}{c} A \longrightarrow B \\ \text{El. Op.} \\ I \longrightarrow E \end{array} \quad \Longrightarrow \quad A = EB$$

Problem

Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

Find elementary matrices E and F so that $C = FEA$.

Solution

Note. The statement of the problem implies that C can be obtained from A by a sequence of two elementary row operations, represented by elementary matrices E and F.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \underset{E}{\to} \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \underset{F}{\to} \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = C$$

where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.Thus we have the sequence $A \to EA \to F(EA) = C$, so $C = FEA$, i.e.,

$$\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$$

You can check your work by doing the matrix multiplication. ■

Inverses of Elementary Matrices

Lemma

Every elementary matrix E is invertible, and E^{-1} is also an elementary matrix (of the same type). Moreover, E^{-1} corresponds to the inverse of the row operation that produces E.

The following table gives the inverse of each type of elementary row operation:

Type	Operation	Inverse Operation
I	Interchange rows p and q	Interchange rows p and q
II	Multiply row p by $k \neq 0$	Multiply row p by $1/k$
III	Add k times row p to row $q \neq p$	Subtract k times row p from row q

Note that elementary matrices of type I are self-inverse.

Inverses of Elementary Matrices

Example

Without using the matrix inversion algorithm, find the inverse of the elementary matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hint. What row operation can be applied to G to transform it to I_4? The row operation $G \rightarrow I_4$ is to add three times row one to row three, and thus

$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check by computing $G^{-1}G$.

Example (continued)

Similarly,

$$E^{-1} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]^{-1} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

and

$$F^{-1} = \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Suppose A is an $m \times n$ matrix and that B can be obtained from A by a sequence of k elementary row operations. Then there exist elementary matrices $E_1, E_2, \ldots E_k$ such that

$$B = E_k(E_{k-1}(\cdots(E_2(E_1A))\cdots))$$

Since matrix multiplication is associative, we have

$$B = (E_kE_{k-1}\cdots E_2E_1)A$$

or, more concisely, $B = UA$ where $U = E_kE_{k-1}\cdots E_2E_1$.

To find U so that $B = UA$, we could find $E_1, E_2, \ldots, E_k$ and multiply these together (in the correct order), but there is an easier method for finding U.

Definition

Let A be an $m \times n$ matrix. We write

$$A \to B$$

if B can be obtained from A by a sequence of elementary row operations. In this case, we call A and B are row-equivalent.

Theorem

Suppose A is an $m \times n$ matrix and that $A \to B$. Then

1. there exists an invertible $m \times m$ matrix U such that $B = UA$;
2. U can be computed by performing elementary row operations on $\left[\begin{array}{c|c} A & I_m \end{array}\right]$ to transform it into $\left[\begin{array}{c|c} B & U \end{array}\right]$;
3. $U = E_k E_{k-1} \cdots E_2 E_1$, where $E_1, E_2, \ldots, E_k$ are elementary matrices corresponding, in order, to the elementary row operations used to obtain B from A.

Problem

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$, and let R be the reduced row-echelon form of A. Find a matrix U so that $R = UA$.

Solution

$$\left[\begin{array}{rrr|rr} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & -3 & -2 & -2 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array}\right] \to \left[\begin{array}{rrr|rr} 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array}\right]$$

Starting with $\left[\begin{array}{c|c} A & I \end{array}\right]$, we've obtained $\left[\begin{array}{c|c} R & U \end{array}\right]$.

Therefore $R = UA$, where

$$U = \begin{bmatrix} 1/3 & 0 \\ 2/3 & -1 \end{bmatrix}.$$

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Example (A Matrix as a product of elementary matrices)

Let

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}.$$

Suppose we do row operations to put A in reduced row-echelon form, and write down the corresponding elementary matrices.

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_3]{}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_4]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_5]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the reduced row-echelon form of A equals I_3. Now find the matrices E_1, E_2, E_3, E_4 and E_5.

Example (continued)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\begin{aligned} (E_5(E_4(E_3(E_2(E_1A))))) &= I \\ (E_5E_4E_3E_2E_1)A &= I \end{aligned}$$

and therefore

$$A^{-1} = E_5E_4E_3E_2E_1$$

Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

$$\begin{aligned} A^{-1} &= E_5E_4E_3E_2E_1 \\ (A^{-1})^{-1} &= (E_5E_4E_3E_2E_1)^{-1} \\ A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \end{aligned}$$

This example illustrates the following result.

Theorem

Let A be an $n \times n$ matrix. Then, A^{-1} exists if and only if A can be written as the product of elementary matrices.

Example (revisited – Matrix inversion algorithm)

$$[\ A \mid I\ ] = \left[\begin{array}{rrr|c} 1 & 2 & -4 & \\ -3 & -6 & 13 & I \\ 0 & -1 & 2 & \end{array}\right]$$

$$E_1 [\ A \mid I\ ] = \left[\begin{array}{rrr|c} 1 & 2 & -4 & \\ 0 & 0 & 1 & E_1 \\ 0 & -1 & 2 & \end{array}\right] = \left[\begin{array}{rrr|rrr} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$E_2E_1 [\ A \mid I\ ] = \left[\begin{array}{rrr|c} 1 & 2 & -4 & \\ 0 & -1 & 2 & E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{rrr|rrr} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

Example (continued)

$$E_3E_2E_1[\,A \mid I\,] = \left[\begin{array}{ccc|c} 1 & 2 & -4 & \\ 0 & 1 & -2 & E_3E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$E_4E_3E_2E_1[\,A \mid I\,] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & -2 & E_4E_3E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$E_5E_4E_3E_2E_1[\,A \mid I\,] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & E_5E_4E_3E_2E_1 \\ 0 & 0 & 1 & \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 6 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{array}\right]$$

$$A^{-1} = E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 2 \\ 6 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

Problem

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

Solution

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \xrightarrow[E_3]{} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[E_4]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Since $E_4E_3E_2E_1A = I$, $A^{-1} = E_4E_3E_2E_1$, and hence

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

Solution (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1}$$

i.e.,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Check your work by computing the product. ■

One result that we have assumed in all our work involving reduced row-echelon matrices is the following.

Theorem (Uniqueness of the Reduced Echelon Form)

If A is an $m \times n$ matrix and R and S are reduced row-echelon forms of A, then $R = S$.

Remark

This theorem ensures that the reduced row-echelon form of a matrix is unique, and its proof follows from the results about elementary matrices.

Smith Normal Form

Definition

If A is an $m \times n$ matrix of rank r, then the matrix $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$ is called the Smith normal form of A.

Theorem

If A is an $m \times n$ matrix of rank r, then there exist invertible matrices U and V of size $m \times m$ and $n \times n$, respectively, such that

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m \times n}$$

Proof.

1. Apply the elementary row operations:

$$[A|I_m] \xrightarrow{\text{e.r.o.}} [\operatorname{rref}(A)\,|U]$$

2. Apply the elementary column operations:

$$\begin{pmatrix} \operatorname{rref}(A) \\ I_n \end{pmatrix} \xrightarrow{\text{e.c.o.}} \begin{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{m\times n} \\ V \end{pmatrix}_{2m\times n}$$

∎

Remark

The elementary column operations above are equivalent to the elementary row operations on the transpose:

$$\left[\operatorname{rref}(A)^T \middle| I_n\right] \xrightarrow{\text{e.r.o.}} \left[\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n\times m} \middle| V^T\right]_{n\times 2m}$$

Problem

Find the decomposition of $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ into the Smith normal form: $A = \widetilde{U} N \widetilde{V}$, where N is the Smith normal form of A and $\widetilde{U}, \widetilde{V}$ are some invertible matrices.

Solution

We have seen that

$$[A|I_2] = \left[\begin{array}{ccc|cc} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1 \end{array}\right] = [\mathrm{rref}(A)|U]$$

Now,

$$\left(\mathrm{rref}(A)^T \;\middle|\; I_3\right) = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] = \left[N^T \middle| V^T\right]$$

Solution (Continued)

Hence, we find $\mathrm{N} = \mathrm{UAV}$, namely,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, since U and V are invertible, we see that

$$\mathrm{A} = \mathrm{U}^{-1}\mathrm{N}\mathrm{V}^{-1},$$

namely,

$$\begin{aligned} \mathrm{A} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} &= \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \widetilde{\mathrm{U}}\mathrm{N}\widetilde{\mathrm{V}}. \end{aligned}$$

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