

Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra

§2-6. Linear Transformations

Le Chen¹

Emory University, 2020 Fall

(last updated on 10/26/2020)



Creative Commons License
(CC BY-NC-SA)

¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Transformations

Finding the Matrix of a Linear Transformation

Composition of Linear Transformations

Rotations in $\mathbb{R}^2$

Reflections in $\mathbb{R}^2$

Reflection in the line

Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **linear transformation** if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let $\vec{x} \in \mathbb{R}^n$. Since T preserves scalar multiplication,

1. $T(0\vec{x}) = 0T(\vec{x})$ implying $T(0) = 0$, so **T preserves the zero vector.**
2. $T((-1)\vec{x}) = (-1)T(\vec{x})$, implying $T(-\vec{x}) = -T(\vec{x})$, so **T preserves the negative of a vector.**

Suppose $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k$ are vectors in $\mathbb{R}^n$ and for some $a_1, a_2, \dots, a_k \in \mathbb{R}$.

$$\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k.$$

$\Downarrow$

3.

$$\begin{aligned} T(\vec{y}) &= T(a_1\vec{x}_1 + a_2\vec{x}_2 + \cdots + a_k\vec{x}_k) \\ &= a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \cdots + a_kT(\vec{x}_k), \end{aligned}$$

i.e., **T preserves linear combinations.**

Problem

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be a linear transformation such that

$$T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}. \text{ Find } T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}.$$

Solution

The only way it is possible to solve this problem is if

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} \text{ is a linear combination of } \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix},$$

i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}.$$

Solution (continued)

To find a and b, solve the system of three equations in two variables:

$$\left[\begin{array}{cc|c} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array} \right]$$

Thus $a = 1$, $b = -2$, and

$$\left[\begin{array}{c} -7 \\ 3 \\ -9 \end{array} \right] = \left[\begin{array}{c} 1 \\ 3 \\ 1 \end{array} \right] - 2 \left[\begin{array}{c} 4 \\ 0 \\ 5 \end{array} \right].$$

Solution (continued)

We now use that fact that linear transformations preserve linear combinations, implying that

$$\begin{aligned} T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T \left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \right) \\ &= T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix} \end{aligned}$$

Therefore, $T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}.$



Problem

Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation such that

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}. \quad \text{Find } T \begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix}.$$

Solution

Final Answer

$$T \begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -3 \end{bmatrix}.$$

Theorem

Every matrix transformation is a linear transformation.

Proof.

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation induced by the $m \times n$ matrix A , i.e., $T(\vec{x}) = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$. Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. Then

$$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T(\vec{x}) + T(\vec{y}),$$

proving that T preserves addition. Also,

$$T(a\vec{x}) = A(a\vec{x}) = a(A\vec{x}) = aT(\vec{x}),$$

proving that T preserves scalar multiplication.

Since T preserves addition and scalar multiplication T is a linear transformation. ■

Example (The Zero Transformation)

If A is the $m \times n$ matrix of zeros, then the transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ induced by A is called the **zero transformation** because for every vector $\vec{x}$ in $\mathbb{R}^n$

$$T(\vec{x}) = A\vec{x} = 0\vec{x} = \vec{0}.$$

Note that the first zero is the matrix A , while the second zero is the zero vector of $\mathbb{R}^m$. The zero transformation is usually written as **$T = 0$** .

Example (The Identity Transformation)

The transformation of $\mathbb{R}^n$ induced by I_n , the $n \times n$ identity matrix, is called the **identity transformation** because for every vector $\vec{x}$ in $\mathbb{R}^n$,

$$T(\vec{x}) = I_n\vec{x} = \vec{x}.$$

The identity transformation on $\mathbb{R}^n$ is usually written as **$1_{\mathbb{R}^n}$** .

Example (Revisited)

Recall $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}$$

Is T a matrix transformation?

Consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, then

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix} = T \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So in this case T is a matrix transformation!

Example (Not all transformations are matrix transformations!)

Consider $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Why is T not a matrix transformation?

Example (continued)

We have $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \vec{x} \in \mathbb{R}^2.$$

Since every matrix transformation is a linear transformation, we consider $T(0)$, where 0 is the zero vector of $\mathbb{R}^2$.

$$T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

violating one of the properties of a linear transformation.

Therefore, T is not a linear transformation, and hence is not a matrix transformation.

Remark

Recall that a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **linear transformation** if it satisfies the following two properties for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and all (scalars) $a \in \mathbb{R}$.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (preservation of addition)
2. $T(a\vec{x}) = aT(\vec{x})$ (preservation of scalar multiplication)

Theorem (Every Linear Transformation is a Matrix Transformation)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then we can find an $n \times m$ matrix A such that

$$T(\vec{x}) = A\vec{x}$$

In this case, we say that T is induced, or determined, by A and we write

$$T_A(\vec{x}) = A\vec{x}$$

Problem

The transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ defined by $T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}$ for each

$\vec{x} \in \mathbb{R}^3$ is another matrix transformation, that is, $T(\vec{x}) = A\vec{x}$ for some matrix A . Can you find a matrix A that works?

Solution

First, since $T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$, we know that A must have size 4×3 . Now consider the product

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix},$$

and try to fill in the values of the matrix.

Solution (continued)

We can deduce from the product that T is induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$



Finding the Matrix of a Linear Transformation

Is there an easier way to find the matrix of T ? For some transformations guess and check will work, but this is not an efficient method. The next theorem gives a method for finding the matrix of T .

Definition

The set of columns $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ of I_n is called the **standard basis of $\mathbb{R}^n$** .

Theorem (Matrix of a Linear Transformation)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then T is a matrix transformation. Furthermore, T is induced by the **unique** matrix

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix},$$

where $\vec{e}_j$ is the j^{th} column of I_n , and $T(\vec{e}_j)$ is the j^{th} column of A .

Corollary

A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Problem

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

for each $\vec{x} \in \mathbb{R}^2$. Find the matrix, A , of T .

Solution

To find A , we must find $T(\vec{e}_1)$ and $T(\vec{e}_2)$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors of $\mathbb{R}^2$.

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 2(0) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 2(1) \\ 0 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The columns $T(\vec{e}_1)$ and $T(\vec{e}_2)$ become the columns of A , so

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix},$$

and $T(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbb{R}^2$. Therefore A is the matrix for T .

Sometimes, T is defined through its actions several concrete vectors.

Problem

Find the matrix A of T where T is given as

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution (continued)

We need to write $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of the vectors provided. First, find x and y such that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Once we find x and y we can compute

$$\begin{aligned} T \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= xT \begin{bmatrix} 1 \\ 1 \end{bmatrix} + yT \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ &= x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} \end{aligned}$$

Solution (continued)

Finding x and y involves solving the following system of equations.

$$\begin{aligned}x &= 1 \\ x - y &= 0\end{aligned}$$

The solution is $x = 1, y = 1$.

Hence, we can find $T(\vec{e}_1)$ as follows.

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

This is the first column of the matrix A . Similarly, we can find $T(\vec{e}_2)$ which will be the second column of A . The resulting matrix is

$$A = \begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$$



Problem

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a transformation defined by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}$.

Is T a linear transformation?

Solution

If T were a linear transformation, then T would be induced by the matrix

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2)] = \left[T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

Since

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix},$$

T is a matrix transformation, and therefore a linear transformation. 

Problem

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a transformation defined by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}$. Is T a linear transformation?

Solution

If T were a linear transformation, then T would be induced by the matrix

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2)] = \left[T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

However,

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x + y \end{bmatrix} \neq \begin{bmatrix} xy \\ x + y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore, T is **not** a linear transformation. 

Composition of Linear Transformations

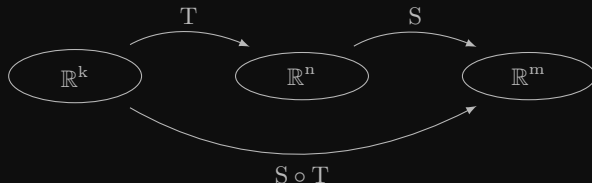
Definition

Suppose $T : \mathbb{R}^k \rightarrow \mathbb{R}^n$ and $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are linear transformations. The **composite** (or composition) of S and T is

$$S \circ T : \mathbb{R}^k \rightarrow \mathbb{R}^m,$$

is defined by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k.$$



Be careful with the order of the transformations! We write $S \circ T$, but it is the transformation T that is applied first, followed by the transformation S .

Theorem

Let $\mathbb{R}^k \xrightarrow{T} \mathbb{R}^n \xrightarrow{S} \mathbb{R}^m$ be linear transformations, and suppose that S is induced by matrix A , and T is induced by matrix B . Then $S \circ T$ is a linear transformation, and $S \circ T$ is induced by the matrix AB .

Example

Let $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be linear transformations defined by

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \quad \text{for all} \quad \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2.$$

Example (continued)

Then S and T are induced by matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

respectively. The composite of S and T is the transformation $S \circ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$(S \circ T) \begin{bmatrix} x \\ y \end{bmatrix} = S \left(T \begin{bmatrix} x \\ y \end{bmatrix} \right),$$

and has matrix (or is induced by the matrix)

$$AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Example (continued)

Therefore the composite of S and T is the linear transformation

$$(S \circ T) \begin{bmatrix} x \\ y \end{bmatrix} = AB \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix},$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.



Remark

Compare this with the composite of T and S which is the linear transformation

$$(T \circ S) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix},$$

for all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$.

Rotations in $\mathbb{R}^2$

The rest part is an application of the linear transform to the study of the rotations in $\mathbb{R}^2$. This is left your motivated students to study by themselves.

Definition

The transformation

$$R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ .

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$R_\theta(\vec{e}_1) = R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix},$$

and

$$R_\theta(\vec{e}_2) = R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

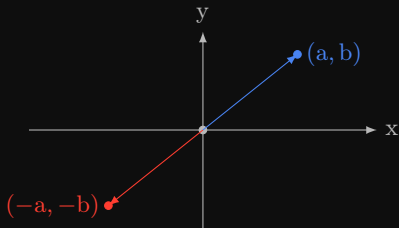
$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π .



We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denotes a **counterclockwise** rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

Solution

First,

$$R_{\frac{\pi}{2}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

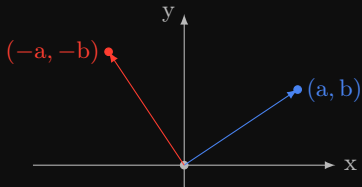


Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so $R_{\pi/2}$ is a matrix transformation.

Reflection in $\mathbb{R}^2$

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

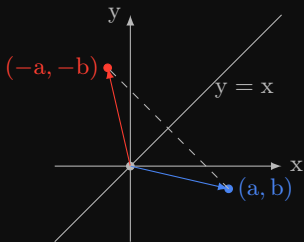
Example

In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



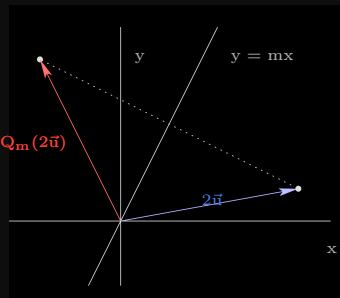
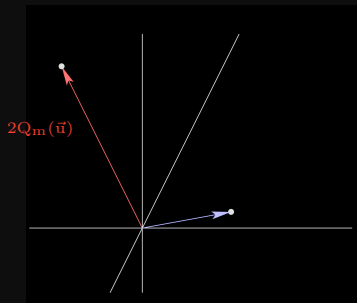
This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Reflection in the line

Example (Reflection in $y = mx$ preserves scalar multiplication)

Let $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.



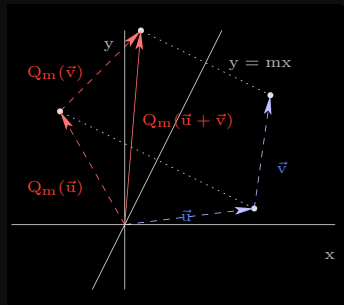
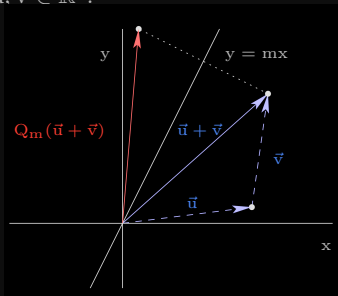
The figure indicates that $Q_m(2\vec{u}) = 2Q_m(\vec{u})$. In general, for any scalar k ,

$$Q_m(kX) = kQ_m(X),$$

i.e., Q_m preserves scalar multiplication.

Example (Reflection in $y = mx$ preserves vector addition)

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$.



The figure indicates that

$$Q_m(\vec{u}) + Q_m(\vec{v}) = Q_m(\vec{u} + \vec{v}),$$

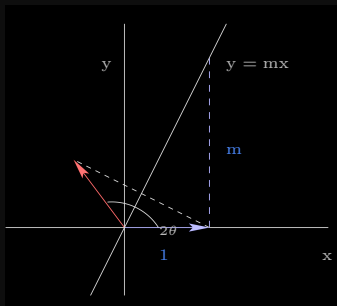
i.e., Q_m preserves vector addition.

Since Q_m preserves addition and scalar multiplication, Q_m is a linear transformation, and hence a matrix transformation.

The matrix that induces Q_m can be found by computing $Q_m(\vec{e}_1)$ and $Q_m(\vec{e}_2)$, where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

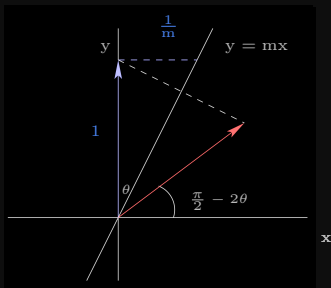
$$Q_m(\vec{e}_1)$$



$$\cos \theta = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin \theta = \frac{m}{\sqrt{1+m^2}}$$

$$Q_m(\vec{e}_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta \\ 2 \sin \theta \cos \theta \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 \\ 2m \end{bmatrix}$$

$$Q_m(\vec{e}_2)$$



$$\cos \theta = \frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \sin \theta = \frac{1}{\sqrt{1+m^2}}$$

$$\begin{aligned} Q_m(\vec{e}_2) &= \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{2} \cos(2\theta) + \sin \frac{\pi}{2} \sin(2\theta) \\ \sin \frac{\pi}{2} \cos(2\theta) - \cos \frac{\pi}{2} \sin(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix} = \begin{bmatrix} 2 \sin \theta \cos \theta \\ \cos^2 \theta - \sin^2 \theta \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 2m \\ m^2 - 1 \end{bmatrix} \end{aligned}$$

Alternatively, we can use the following relation to find Q_m :

$$Q_m = R_\theta \circ Q_0 \circ R_{-\theta}$$

$$R_\theta \sim \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad Q_0 \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad R_{-\theta} \sim \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

Then multiply these three matrices ...

The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Problem (Multiple Actions)

Find the rotation or reflection that equals reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

Solution

Let Q_0 denote the reflection in the x-axis, and $R_{\frac{\pi}{2}}$ denote the rotation through an angle of $\frac{\pi}{2}$. We want to find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution

Hence $R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a **reflection** matrix.

How do we know this?

Solution (continued)

Compare BA to

$$Q_m = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Now, since $1-m^2=0$, we know that $m=1$ or $m=-1$. But $\frac{2m}{1+m^2}=1>0$, so $m>0$, implying $m=1$.

Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y=x$. 

Problem (Reflection followed by Reflection)

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y -axis.

Solution

We must find the matrix for the transformation $Q_Y \circ Q_{-1}$.

Q_{-1} is induced by

$$A = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and Q_Y is induced by

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $Q_Y \circ Q_{-1}$ is induced by BA .

Solution (continued)

$$\mathbf{BA} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

What transformation does $\mathbf{BA}$ induce?

Rotation through an angle θ such that

$$\cos \theta = 0 \quad \text{and} \quad \sin \theta = -1.$$

Therefore, $\mathbf{Q}_Y \circ \mathbf{Q}_{-1} = \mathbf{R}_{-\frac{\pi}{2}} = \mathbf{R}_{\frac{3\pi}{2}}$.



Remark (Summary)

In general,

- The composite of two rotations is a rotation

$$R_\theta \circ R_\eta = R_{\theta+\eta}$$

- The composite of two reflections is a rotation.

$$Q_m \circ Q_n = R_\theta$$

where θ is $2\times$ the angle between lines $y = mx$ and $y = nx$.

- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$