Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-7. LU Factorization

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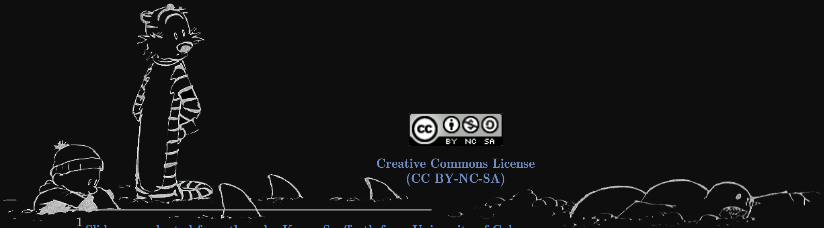


1 Slides are adapted from those by Karen Seyffarth from University of Calgary.

LU Factorization

Why do we need LU Factorization?

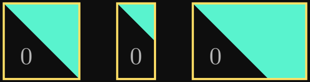
Finding the LU

Multiplier Method

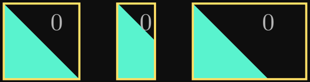
LU Factorization

Definition

A matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries below the main diagonal equal 0.



A lower triangular matrix is defined similarly, as a matrix for which all entries above the main diagonal are equal to zero.



An LU factorization of a matrix A is written

$$A = LU$$

where L is lower triangular matrix and U is upper triangular.

We often require either L or U to have only 1's on the main diagonal.

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

Why do we need LU Factorization?

The LU factorization often helps to quickly solve equations of the form $A\vec{x} = \vec{b}$.

Suppose we wish to find all solutions $\vec{x}$ to the system $A\vec{x} = B$. The LU factorization of A can assist in this process.

Consider the following reduction:

$$\begin{aligned} A\vec{x} &= B \\ (LU)\vec{x} &= B \\ L(U\vec{x}) &= B \\ L\vec{y} &= B \end{aligned}$$

Therefore, if we can solve $L\vec{y} = B$ for $\vec{y}$, then all that remains is to solve $U\vec{x} = \vec{y}$ for $\vec{x}$.

Example

Find all solutions to

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 10 & 5 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Solution

Using a method of your choice, verify that the LU factorization of A gives

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

The solution is $\vec{y} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

Now we solve $U\vec{x} = \vec{y}$.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

Solution (continued)

Multiplying and solving (or finding the reduced row-echelon form), the general solution is given by

$$\vec{x} = \begin{bmatrix} -12 \\ 2 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 13 \\ -3 \\ -2 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

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Finding the LU Factorization

An LU factorization can be found for a matrix A provided that the row-echelon form of A can be calculated without interchanging rows. In this case, we call that A can be lower reduced.

Example

Determine if the LU factorization of A exists, and if so, find it.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution

Because the row-echelon form can be obtained without interchanging rows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_3 + r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

the LU factorization exists, or A can be lower reduced.

Solution (continued)

We proceed to finding L and U. Assign variables to the unknown entries and multiply.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

Solving each entry will give us values for the unknown entries.

Solution (continued)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} \mathrm{a} & \mathrm{d} & \mathrm{e} \\ \mathrm{ax} & \mathrm{dx}+\mathrm{b} & \mathrm{ex}+\mathrm{f} \\ \mathrm{ay} & \mathrm{dy}+\mathrm{bz} & \mathrm{ey}+\mathrm{fz}+\mathrm{c} \end{bmatrix}$$

We see easily that $\mathrm{a}=1, \mathrm{d}=1$, and $\mathrm{e}=2$. Continuing to solve the first column gives $\mathrm{x}=2, \mathrm{y}=1$. The other values are calculated as follows.

$$\begin{aligned} \mathrm{dx}+\mathrm{b} &= 3 \\ (1)(2)+\mathrm{b} &= 3 \\ \mathrm{b} &= 1 \end{aligned}$$

$$\begin{aligned} \mathrm{ex}+\mathrm{f} &= 0 \\ (2)(2)+\mathrm{f} &= 0 \\ \mathrm{f} &= -4 \end{aligned}$$

$$\begin{aligned} \mathrm{dy}+\mathrm{bz} &= 0 \\ (1)(1)+(1)\mathrm{z} &= 0 \\ \mathrm{z} &= -1 \end{aligned}$$

$$\begin{aligned} \mathrm{ey}+\mathrm{fz}+\mathrm{c} &= 5 \\ (2)(1)+(-4)(-1)+\mathrm{c} &= 5 \\ \mathrm{c} &= -1 \end{aligned}$$

Solution (continued)

Therefore,

$$
\begin{array}{ccc}
L & & U \\
\| & & \| \\
\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} & & \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \\
\| & & \| \\
\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} & & \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}
\end{array}
$$

You should multiply these and check that they equal A!

Solution (continued)

If you want the diagonal terms of U to be all 1's:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 1 & 0 & -0 \\ 2 & 1 & -0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ -0 & -0 & 1 \end{bmatrix}$$

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Multiplier Method

The following process for finding L and U, called the multiplier method, can be more efficient.

Example

Find the LU factorization of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution

First, write A as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$||$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution (continued)

To do so, we use row operations to remove the entries of A below the main diagonal. For every operation we apply to A (the matrix on the right), we apply the inverse operation to the identity matrix (on the left). This ensures the product remains the same.

The first step is to add (-2) times the first row of A to the second row. To preserve the product, add (2) times the second column to the first column, for the matrix on the left.

$$\|$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\|$$

$$c_1 + 2c_2 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} r_2 - 2r_1 \to r_2$$

Solution (continued)

We proceed in the same way.

$$c_1 + c_3 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} r_3 - r_1 \to r_3$$

$$c_2 - c_3 \to c_2 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix} r_3 + r_2 \to r_3$$

At this point we have a lower triangular matrix L on the left, and an upper triangular matrix U on the right so we are done. You can (and should!) check that this product equals A.

If you want the diagonal terms of U to be all 1's:

$$-1 \times c_3 \to c_3 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} - 1 \times r_3 \to r_3$$

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Problem

Use the multiplier method to verify the LU factorization for

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

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The multiplier method can be simplified using so-called LU-Algorithm.
See Examples 2.7.2 – 2.7.4 on the book.